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CURVED FINITE ELEMENTS OF CLASS C1 : IMPLEMENTATION AND NUMERICAL EXPERIMENTS
Part 1 : Construction and numerical tests of the interpolation properties

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Juin 1992

RR-1718

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CURVED FINITE ELEMENTS OF CLASS $\mathcal{C}^1$: IMPLEMENTATION AND NUMERICAL EXPERIMENTS

(*)(**)

Part 1 : Construction and numerical tests of the interpolation properties

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Abstract :

The aim of this work is to show how to implement some of the methods of curved finite elements of class $\mathcal{C}^1$ introduced and analyzed in a previous work by the first author. These methods use curved finite elements which are $\mathcal{C}^1$-compatible with the Argyris triangle. The careful description of the implementation is completed by some numerical experiments which show the very high degree of accuracy of the associated interpolation. Subsequently, in a second part, we will show the efficiency of such methods for solving thin plate or thin shell problems set on curved boundary domains.

ELEMENTS FINIS COURBES DE CLASSE $\mathcal{C}^1$: IMPLEMENTATION ET TESTS NUMERIQUES

Partie 1 : Construction et tests numériques des propriétés d'interpolation

Résumé :

Le but de ce travail est de montrer comment implémenter certaines des méthodes d'éléments finis courbes de classe $\mathcal{C}^1$ introduites et analysées dans un travail précédent du premier auteur. Ces méthodes utilisent des éléments finis courbes qui sont $\mathcal{C}^1$-compatibles avec le triangle d'Argyris. La description détaillée de l'implémentation est complétée par des tests numériques qui montrent le très haut degré de précision de l'interpolation correspondante. Ultérieurement, dans la seconde partie, nous illustrerons l'efficacité de ces méthodes pour la résolution de problèmes de plaques et de coques minces formulés sur des domaines à frontière curviligne.

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INTRODUCTION

The conforming approximation of thin plate problems with curved boundaries requires the use of curved $\mathcal{C}^1$-finite elements which are compatible with some classical $\mathcal{C}^1$-straight finite elements. A similar situation occurs for the conforming approximation of thin shells or for the conforming approximation of junctions between thin shells (see [1]).

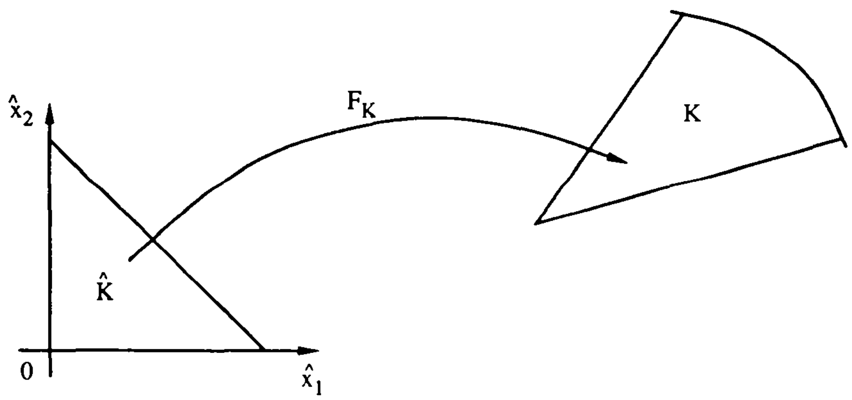
We have introduced and analyzed such curved elements in [2,3] so that they are $\mathcal{C}^1$-compatible with the ARGYRIS triangle [4,5,6]. These methods are particularly interesting to realize a very accurate approximation of thin plate or thin shell problems [2,3].

In this paper, we record the mains steps of the construction of curved finite elements which are $\mathcal{C}^1$-compatible with the ARGYRIS triangle. and then we give a very detailed description of how to implement such methods. More precisely, paragraph 1 is dedicated to the construction of the approximated domain Ω_h of the given curved boundary domain Ω while paragraph 2 contains the main steps of the definition of the considered curved $\mathcal{C}^1$ finite element. Next, in paragraph 3, we detail the matrix decompositions which allow to realize the interpolation modules. Finally, in paragraph 4 we report some numerical experiments which prove the efficiency and the very high accuracy of such interpolation methods.

In addition it is worth to mention that all the relations have been checked by using the formal computational tools provided by Mathematica [7].

Summary of the construction of these curved $\mathcal{C}^1$-elements

Let v be the function to be interpolated on the curved triangle K. The construction of



these curved $\mathcal{C}^1$-elements takes several steps :

i) triangulation of the curved boundary domain Ω
ii) interpolation of the curved triangular sides located on the boundary
iii) definition of the mapping F_K ;
iv) computation of the set of values of the degrees of freedom $\Sigma_K(v)$ of function v ;
v) from $\Sigma_K(v)$, computation of the set of values of the degrees of freedom $\hat{\Delta}_K(v)$;
vi) from the set $\hat{\Delta}_K(v)$, computation of the interpolate function $\hat{w}$;
vii) computation of the interpolate function $\pi_K v = w = \hat{w} \circ F_K^{-1}$.

Notations

For more details concerning curved $\mathcal{C}^1$ finite elements and for notations we refer to [2,3] and, more generally, finite element notations are those used by [5,6].

Contents

1 APPROXIMATION OF THE CURVED BOUNDARY DOMAIN Ω

This paragraph records some useful results obtained in [2,3].

1.1 "Exact" triangulation of the domain Ω

Let Ω be a plane domain with a curved boundary Γ. We realize partitions of this domain by using regular families of triangulations $\mathcal{T}_h$ including (see Fig. 1.1.1) :
i) straight triangles, on the one hand ;
ii) triangles with one curved side located on Γ, on the other hand.

In this way, triangulations $\mathcal{T}_h$ can be splitted as follows : $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ where $\mathcal{T}_h^1$ collects all the straight triangles K while $\mathcal{T}_h^2$ collects all the triangles K_c with one curved side on Γ.

According to Figure 1.1.1, we note $\tilde{K}$ the straight triangles associated to a one-curved side triangle K_c.

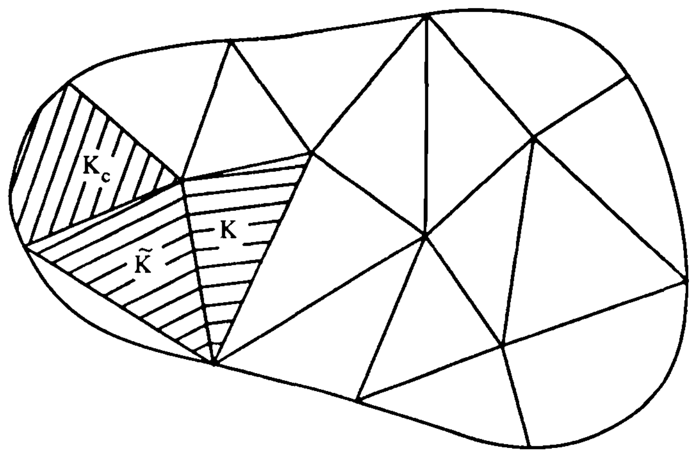


Figure 1.1.1 : "Exact" triangulation of the domain Ω made up of straight triangles K and one-curved side triangles K_c.

1.2 "Approximate" triangulation of the domain Ω

Any one-curved side triangle K_c is approximate by a triangle K. In this way, the curved side of K_c is approximated by an arc γ_h (see Figure 1.2.1). Then the union of all straight triangles $K \in \mathcal{T}_h^1$ and of all one-curved side triangles $K \in \mathcal{T}_h^2$ give the approximate domain Ω_h.

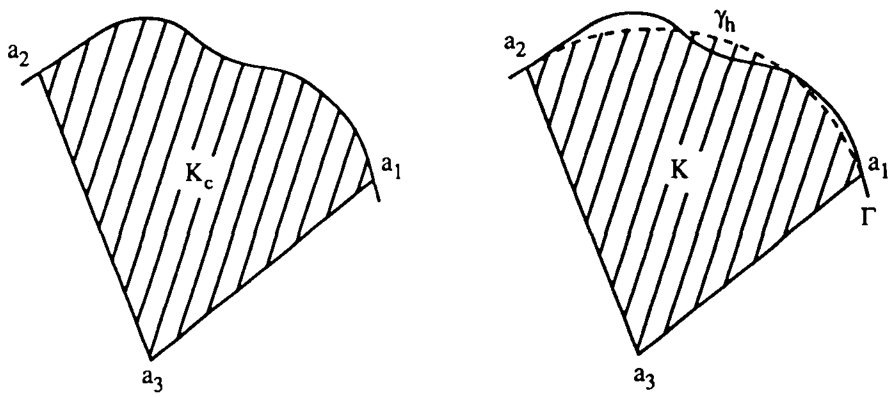


Figure 1.2.1 : "Exact" and "approximate" one-curved side triangles K_c and K

Such a construction can be splitted in two steps :

Step 1 : Interpolation of the curved side a_1a_2 of the triangle K_c. Assume that $\mathbb{R}^2$ is referred to an orthonormal system $(O, \vec{e}_1, \vec{e}_2)$ whose associate coordinates are noted

(x_1, x_2) and that Ω is a bounded domain. Moreover, assume that the boundary Γ can be subdivided in a finite number of arcs, each of them having a sufficiently smooth parametric representation of the following type :

$$x_1 = \chi_1(s), \;\; x_2 = \chi_2(s), \;\; \underline{s} \leq s \leq \bar{s}. \tag{1.2.1}$$

Subsequently, we also use another parameterization of the arc $\widehat{a_1 a_2}$:

$$x_1 = \psi_1(\hat{x}_2), \;\; x_2 = \psi_2(\hat{x}_2), \;\; 0 \leq \hat{x}_2 \leq 1, \tag{1.2.2}$$

where

$$\psi_\alpha(\hat{x}_2) = \chi_\alpha(\underline{s} + (\bar{s} - \underline{s})\hat{x}_2), \;\; \alpha = 1, 2. \tag{1.2.3}$$

Then, every component $\psi_\alpha(\hat{x}_2)$ is interpolated by a polynomial function $\psi_{\alpha h}$ of degree $n \geq 2$, so that

$$\psi_{\alpha h}(0) = \psi_\alpha(0), \;\; \psi_{\alpha h}(1) = \psi_\alpha(1), \;\; \alpha = 1, 2. \tag{1.2.4}$$

Thus, for $n \geq 2$, we get :

$$\psi_{\alpha h}(\hat{x}_2) = x_{\alpha 1} + (x_{\alpha 2} - x_{\alpha 1})\hat{x}_2 + \hat{x}_2(1 - \hat{x}_2)P_{n-2;\alpha}(\hat{x}_2) \tag{1.2.5}$$

where $P_{n-2;\alpha}(\hat{x}_2)$ refers to polynomials of degree $n - 2$ with respect to $\hat{x}_2$. These relations (1.2.5) define the approximate arc γ_h. With the notations of Fig. 1.2.2, observe that relation (1.2.5) can be geometrically interpretated as follows

$$\overrightarrow{OP_h} = \overrightarrow{O\tilde{P}} + \overrightarrow{\tilde{P}P_h} \tag{1.2.6}$$

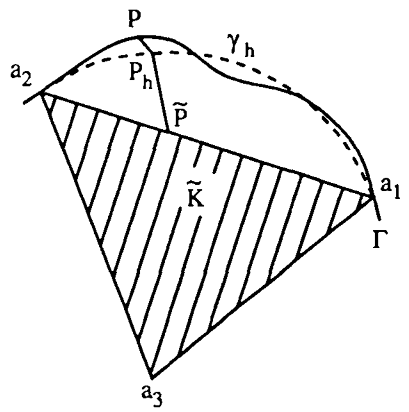


Figure 1.2.2 : Construction of the approximate arc γ_h

Step 2 : Definition of the application $F_K : \hat{K} \longrightarrow K$. Let $\hat{K}$ be a given reference triangle, for instance the unit right-angled triangle $\hat{a}_1\hat{a}_2\hat{a}_3$ with $\hat{a}_1 = (1, 0)$, $\hat{a}_2 = (0, 1)$, $\hat{a}_3 = (0, 0)$. Then, to any point $\hat{M}$ of the reference triangle $\hat{K}$, the application F_K associates the point M_h :

$$\overrightarrow{OM_h} = F_{K1}(\hat{x}_1, \hat{x}_2)\vec{e}_1 + F_{K2}(\hat{x}_1, \hat{x}_2)\vec{e}_2 \tag{1.2.7}$$

where the functions $F_{K\alpha}$, $\alpha = 1,2$ are defined for $n \geq 2$ by the relations

$$(1.2.8) \qquad \left\{ \begin{array}{l} F_{K\alpha}(\hat{x}_1, \hat{x}_2) = x_{\alpha 3} + (x_{\alpha 1} - x_{\alpha 3})\hat{x}_1 + (x_{\alpha 2} - x_{\alpha 3})\hat{x}_2 \\ \qquad + \dfrac{1}{2}\, \hat{x}_1 \hat{x}_2 [P_{n-2;\alpha}(1 - \hat{x}_1) + P_{n-2;\alpha}(\hat{x}_2)]. \end{array} \right.$$

These components are symmetrical with respect to $\hat{x}_1$ and $\hat{x}_2$, and they can be geometrically interpretated in the vectorial form (see Figure 1.2.3) :

$$(1.2.9) \qquad \overrightarrow{OM_h} = \overrightarrow{O\tilde{M}} + \frac{1}{2}\ \overrightarrow{\tilde{M}M_h^1} + \frac{1}{2}\ \overrightarrow{\tilde{M}M_h^2}$$

The properties of this application $F_K : \hat{K} \longrightarrow K$ are detailed in [2].

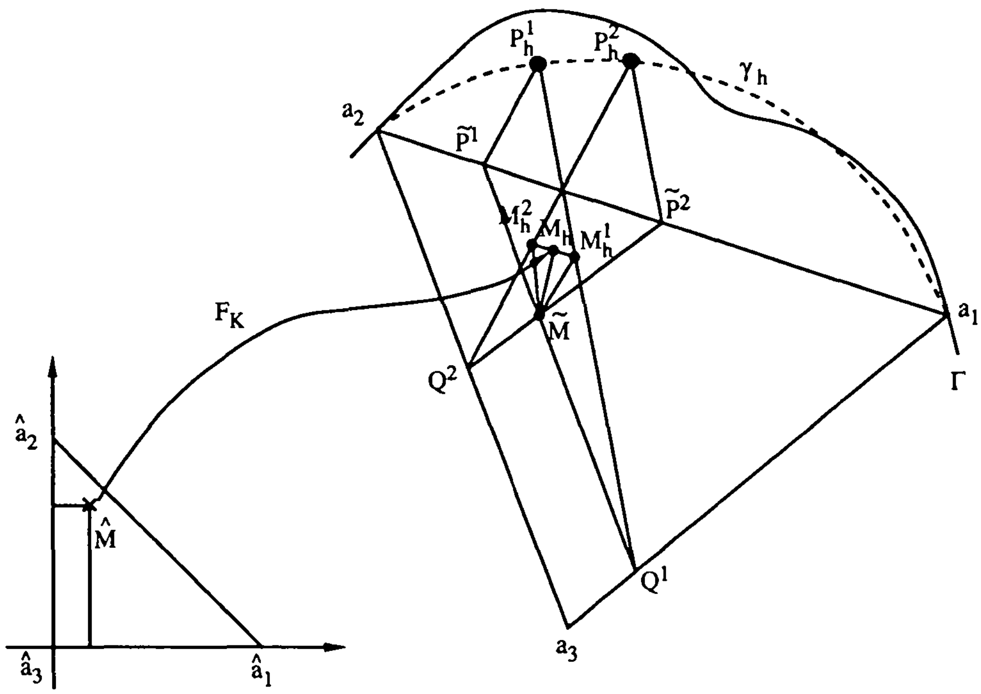


Figure 1.2.3 : Transformation $\tilde{M} \longrightarrow M_h$ (triangles $Q^1\tilde{M}M_h^1$ and $Q^1\tilde{P}^1P_h^1$ are homothetic in the ratio $\hat{x}_2/(1 - \hat{x}_1)$ and triangles $Q^2\tilde{M}M_h^2$ and $Q^2\tilde{P}^2P_h^2$ are homothetic in the ratio $\hat{x}_1/(1 - \hat{x}_2)$).

1.3 Examples

Hereunder, we give two examples of applications $\psi_{\alpha h}$ (see (1.2.5)) which define an approximate arc γ_h interpolating a given arc a_1a_2. These examples correspond to interpolations by polynomials of degree $n = 3$ or 5 which are in practice the most interesting. For each case, we indicate the expression of the applications $\psi_{\alpha h}$ from which we deduce the expression of the components $F_{K\alpha}$, $\alpha = 1,2$. Afterwards, these expressions are permanently used ; in order to make distinction between them, we note F_K the one which is associated to the example 1.3.1 (i.e., $F_K \in (P_3)^2$) and F_K^* the one which corresponds to example 1.3.2 (i.e., $F_K^* \in (P_5)^2$).

Example 1.3.1 : Construction of the approximate arc γ_h by using polynomials of degree 3 (Figure 1.3.1)

Expressions (1.2.5) give

$$(1.3.1)\qquad \begin{cases} \psi_{\alpha h}(\hat{x}_2) = x_{\alpha 1} + (x_{\alpha 2} - x_{\alpha 1})\hat{x}_2 + \hat{x}_2(1-\hat{x}_2)\{[2(x_{\alpha 2} - x_{\alpha 1}) \\ \qquad - (\bar{s} - \underline{s})(\chi'_\alpha(\underline{s}) + \chi'_\alpha(\bar{s}))]\hat{x}_2 + x_{\alpha 1} - x_{\alpha 2} + (\bar{s} - \underline{s})\chi'_\alpha(\underline{s})\} \end{cases}$$

so that from (1.2.8), we obtain

$$(1.3.2)\qquad \begin{cases} F_{K\alpha}(\hat{x}_1,\hat{x}_2) = x_{\alpha 3} + (x_{\alpha 1} - x_{\alpha 3})\hat{x}_1 + (x_{\alpha 2} - x_{\alpha 3})\hat{x}_2 + \dfrac{1}{2}\,\hat{x}_1\hat{x}_2\{[2(x_{\alpha 2} - x_{\alpha 1}) \\ \qquad - (\bar{s} - \underline{s})(\chi'_\alpha(\underline{s}) + \chi'_\alpha(\bar{s}))][\hat{x}_2 - \hat{x}_1] + (\bar{s} - \underline{s})[\chi'_\alpha(\underline{s}) - \chi'_\alpha(\bar{s})]\} \end{cases}$$

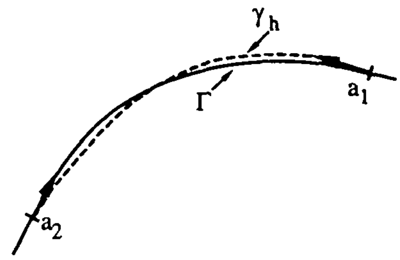


Figure 1.3.1 : Boundary interpolation with polynomials of degree 3

Example 1.3.2 : Construction of the approximate arc γ_h^* by using polynomials of degree 5

We interpolate the functions ψ_α, $\alpha = 1,2$, on the interval $[0,1]$ by using Hermite polynomials of degree 5. The degrees of freedom of the interpolation are the values of the functions $\psi_\alpha, \psi'_\alpha$ and ψ''_α at points $\hat{x}_2 = 0$ or $\hat{x}_2 = 1$, i.e.,

$$(1.3.3)\qquad \begin{cases} x_{\alpha 1} = \psi_\alpha(0) = \chi_\alpha(\underline{s}), \quad x_{\alpha 2} = \psi_\alpha(1) = \chi_\alpha(\bar{s}), \\ \psi_\alpha^{(\ell)}(0) = (\bar{s} - \underline{s})^\ell \chi_\alpha^{(\ell)}(\underline{s}), \quad \psi_\alpha^{(\ell)}(1) = (\bar{s} - \underline{s})^\ell \chi_\alpha^{(\ell)}(\bar{s}), \quad \ell = 1,2. \end{cases}$$

From the expression (1.2.5) we obtain (we recall that all the results related to the example 1.3.2 are indexed with a star (*) in order to make the distinction with similar results related to the example 1.3.1)

$$(1.3.4)\qquad \psi^*_{\alpha h}(\hat{x}_2) = x_{\alpha 1} + (x_{\alpha 2} - x_{\alpha 1})\hat{x}_2 \;+\; \hat{x}_2(1-\hat{x}_2)[\beta_{\alpha 3}(\hat{x}_2)^3 + \beta_{\alpha 2}(\hat{x}_2)^2 + \beta_{\alpha 1}\hat{x}_2 + \beta_{\alpha 0}]$$

where the coefficients $\beta_{\alpha\ell}$, $\ell = 0,1,2,3$ are given by the relations

$$
(1.3.5)\quad \begin{cases}
\beta_{\alpha 0} = x_{\alpha 1} - x_{\alpha 2} + (\bar{s} - \underline{s})\chi'_\alpha(\underline{s}) \\
\beta_{\alpha 1} = x_{\alpha 1} - x_{\alpha 2} + (\bar{s} - \underline{s})\chi'_\alpha(\underline{s}) + \dfrac{(\bar{s} - \underline{s})^2}{2}\,\chi''_\alpha(\underline{s}) \\
\beta_{\alpha 2} = 9(x_{\alpha 2} - x_{\alpha 1}) - (\bar{s} - \underline{s})[5\chi'_\alpha(\underline{s}) + 4\chi'_\alpha(\bar{s})] - \dfrac{(\bar{s} - \underline{s})^2}{2}\,[2\chi''_\alpha(\underline{s}) - \chi''_\alpha(\bar{s})] \\
\beta_{\alpha 3} = 6(x_{\alpha 1} - x_{\alpha 2}) + 3(\bar{s} - \underline{s})[\chi'_\alpha(\underline{s}) + \chi'_\alpha(\bar{s})] + \dfrac{(\bar{s} - \underline{s})^2}{2}\,[\chi''_\alpha(\underline{s}) - \chi''_\alpha(\bar{s})]
\end{cases}
$$

From the expressions (1.2.8) and (1.3.4), we deduce

$$
(1.3.6)\quad \begin{cases}
F^*_{K\alpha}(\hat{x}_1, \hat{x}_2) = x_{\alpha 3} + (x_{\alpha 1} - x_{\alpha 3})\hat{x}_1 + (x_{\alpha 2} - x_{\alpha 3})\hat{x}_2 \\
\qquad + \dfrac{1}{2}\,\hat{x}_1\hat{x}_2[\beta_{\alpha 3}(\hat{x}_2)^3 + \beta_{\alpha 2}(\hat{x}_2)^2 + \beta_{\alpha 1}\hat{x}_2 + \beta_{\alpha 0} \\
\qquad + \tilde{\beta}_{\alpha 3}(\hat{x}_1)^3 + \tilde{\beta}_{\alpha 2}(\hat{x}_1)^2 + \tilde{\beta}_{\alpha 1}\hat{x}_1 + \tilde{\beta}_{\alpha 0}]
\end{cases}
$$

where the coefficients $\beta_{\alpha\ell}$, $\ell = 0,1,2,3$ are given by relation (1.3.5) and where coefficients $\tilde{\beta}_{\alpha\ell}$, $\ell = 0,1,2,3$ are given by

$$
(1.3.7)\quad \begin{cases}
\tilde{\beta}_{\alpha 0} = x_{\alpha 2} - x_{\alpha 1} - (\bar{s} - \underline{s})\chi'_\alpha(\bar{s}) \\
\tilde{\beta}_{\alpha 1} = x_{\alpha 2} - x_{\alpha 1} - (\bar{s} - \underline{s})\chi'_\alpha(\bar{s}) + \dfrac{(\bar{s} - \underline{s})^2}{2}\,\chi''_\alpha(\bar{s}) \\
\tilde{\beta}_{\alpha 2} = 9(x_{\alpha 1} - x_{\alpha 2}) + (\bar{s} - \underline{s})[5\chi'_\alpha(\bar{s}) + 4\chi'_\alpha(\underline{s})] - \dfrac{(\bar{s} - \underline{s})^2}{2}\,[2\chi''_\alpha(\bar{s}) - \chi''_\alpha(\underline{s})] \\
\tilde{\beta}_{\alpha 3} = 6(x_{\alpha 2} - x_{\alpha 1}) - 3(\bar{s} - \underline{s})[\chi'_\alpha(\bar{s}) + \chi'_\alpha(\underline{s})] + \dfrac{(\bar{s} - \underline{s})^2}{2}\,[\chi''_\alpha(\bar{s}) - \chi''_\alpha(\underline{s})]
\end{cases}
$$

2 DEFINITION OF CURVED FINITE ELEMENTS, $\mathcal{C}^1$-COMPATIBLE WITH ARGYRIS TRIANGLE

In this paragraph we record the definitions of **two curved finite elements** which have a connection of class $\mathcal{C}^1$ with the classical ARGYRIS triangle.

i) the **first** corresponds to the interpolation of the boundary considered in the example 1.3.1. This interpolation is realized by using polynomials of degree 3 and it turns out to be sufficient for the approximation of fourth-order problems with homogeneous Dirichlet boundary conditions (see [3]) ;

ii) the **second** corresponds to the interpolation of the boundary described in the example 1.3.2. This interpolation is realized by using polynomials of degree 5 and it can be used for more general boundary conditions (see [3, Remark 3.1]).

Both constructions are fully detailed in [2,3] where, in addition, one can find necessary justifications and corresponding interpolation error estimates. The implementation of these methods is detailed in matrix form in paragraph 3.

2.1 Basic principles

We consider separately **essential** and **desirable** conditions.

Essential conditions :

According to Figure 1.2.1, the connections between ARGYRIS triangles and curved elements are made along the straight sides a_3a_1 and a_3a_2. Let (K, P_K, Σ_K) be the approximate curved triangle, its associated functional space and its corresponding set of degrees of freedom. To obtain a connection of class $\mathcal{C}^1$, it is essential to satisfy (see Figure 2.1.1) :

(2.1.1) $\left\{\begin{array}{l}\text{the degrees of freedom of the curved finite elements related to the sides} \\ a_3a_1 \text{ and } a_3a_2 \text{ are identical to those of the ARGYRIS triangle ;}\end{array}\right.$

(2.1.2) $\left\{\begin{array}{l}\text{the traces } p|_{[a_3,a_\alpha]},\ \alpha = 1,2 (\text{resp.} Dp(.)(a_2 - c_2)|_{[a_3,a_1]}\ ;\ Dp(.)(a_1 - c_1)|_{[a_3,a_2]}) \\ \text{of the functions } p \in P_K \text{ defined on the curved triangle } K\text{, are one-variable} \\ \text{polynomial of degree 5 (resp. 4), entirely determined by the degrees of} \\ \text{freedom related to the sides } a_3a_\alpha,\ \alpha = 1,2.\end{array}\right.$

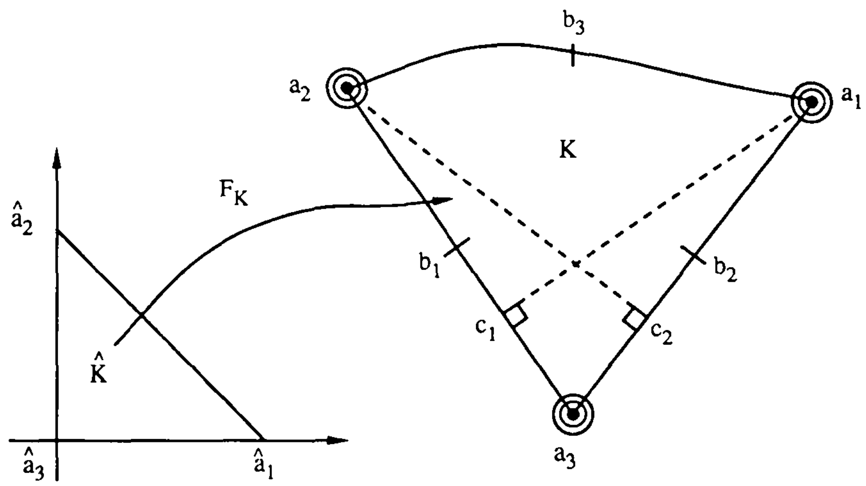


Figure 2.1.1 : The approximate curved triangle K

Desirable conditions : The application F_K, introduced in paragraph 1, associates the curved triangle K to the reference triangle $\hat{K}$.

We use this application F_K as well to associate at any function v defined on the triangle K, one function $\hat{v}$ defined on the triangle $\hat{K}$, i.e.,

(2.1.3) $$v = \hat{v} \circ (F_K)^{-1} \ ;\ \hat{v} = v \circ F_K.$$

Consequently, it is "desirable" that the following condition is satisfied :

$$(2.1.4)\qquad \left\{ \begin{array}{l} \text{To any function } p \in P_K, \text{ defined on the curved triangle } K, \text{ the} \\ \text{correspondence (2.1.3) associates a } \textbf{polynomial} \text{ function } \hat{p} = p \circ F_K. \end{array} \right.$$

This condition (2.1.4) is convenient to study the approximation error, to study the effect of numerical integration scheme and to take into account the boundary conditions. But this condition leads to the definition of reference finite elements which are most complicated than those associated to corresponding straight side finite elements. Indeed, with notations of Figure 2.1.1, we obtain for $a \in [a_3a_1]$:

$$(2.1.5)\qquad \left\{ \begin{array}{l} \dfrac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{a}) = D\hat{p}(\hat{a})\vec{e}_2 = Dp(a)DF_K(\hat{a})\vec{e}_2 = Dp(a)\,\dfrac{\partial F_K}{\partial \hat{x}_2}(\hat{a}) \\[2ex] \qquad = < \dfrac{\partial F_K}{\partial \hat{x}_2}(\hat{a}),\ \dfrac{a_1 - a_3}{|a_1 - a_3|^2}\, Dp(a)(a_1 - a_3) \ +\ \dfrac{a_2 - c_2}{|a_2 - c_2|^2}\, Dp(a)(a_2 - c_2) > \end{array} \right.$$

where $< .,. >$ denotes the usual scalar product in $\mathbb{R}^2$. Likewise for $a \in [a_3a_2]$, we get :

$$(2.1.6)\quad \frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{a}) = < \frac{\partial F_K}{\partial \hat{x}_1}(\hat{a}),\ \frac{a_2 - a_3}{|a_2 - a_3|^2}\, Dp(a)(a_2 - a_3) \ +\ \frac{a_1 - c_1}{|a_1 - c_1|^2}\, Dp(a)(a_1 - c_1) >.$$

Relations (2.1.5) and (2.1.6) prove that $\dfrac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{a})$ (resp. $\dfrac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{a})$) is a polynomial of degree $n+3$ with respect to $\hat{x}_1$ (resp. $\hat{x}_2$) for any $\hat{a} \in [\hat{a}_3\hat{a}_1]$ (resp. $\hat{a} \in [\hat{a}_3\hat{a}_2]$), with $n = 3$ or 5 depending on whether F_K is of degree 3 or 5(see examples 1.3.1 and 1.3.2). Thus to $\hat{K}$ we have to associate a finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ such that

$$(2.1.7)\qquad \hat{P}_K \subset P_{n+4} \quad (n = 3 \text{ or } 5)\ ;\ \hat{P}_K = \{\hat{p} : \hat{K} \longrightarrow \mathbb{R}\ ;\ \hat{p} = p \circ F_K,\ \ p \in P_K\}.$$

2.2 Definition of curved finite elements C^1-compatible with ARGYRIS triangle for $F_K \in (P_3)^2$

In all this section we consider the application F_K defined in the example 1.3.1. Then, relation (2.1.7) gives $\hat{P}_K \subset P_7$.

2.2.1 The basic finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$

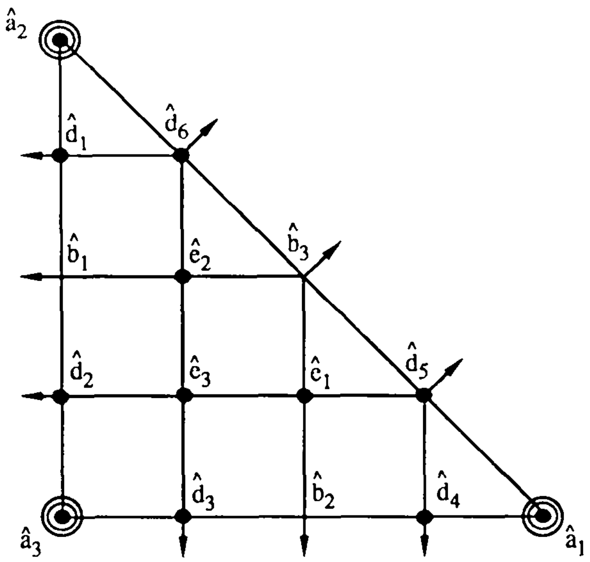
This finite element is described in Figure 2.2.1 and we refer the reader to [8] for more details. Corresponding basis functions are given by relation (3.1.43).

2.2.2 Construction of the interpolating function $v \longrightarrow \pi_K v$

Now we use this basic finite element in order to associate an interpolate function $\pi_K v$ to any function $v \in C^2(K)$. This construction takes three steps :

Step 1 : Definition of the set Σ_K of the degrees of freedom of the curved element. Set (see Figures 2.2.1 and 2.2.2)

$$(2.2.1)\qquad \left\{ \begin{array}{lcl} a_i & = & F_K(\hat{a}_i),\ \ b_i = F_K(\hat{b}_i),\ \ e_i = F_K(\hat{e}_i),\ \ i = 1,2,3\ ; \\[2ex] c_\alpha & = & \text{orthogonal projection of } a_\alpha \text{ on the side } a_3a_\alpha,\ \ \alpha = 1,2,. \end{array} \right.$$



$\hat{K} =$ unit right-angled triangle ; $\hat{P} = P_7$; $dim\hat{P} = 36$

$$\left\{\begin{array}{l} \hat{\Sigma}(\hat{p}) = [\hat{p}(\hat{a}_i),\ \dfrac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{a}_i),\ \dfrac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{a}_i),\ \dfrac{\partial^2 \hat{p}}{\partial \hat{x}_1^2}(\hat{a}_i),\ \dfrac{\partial^2 \hat{p}}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_i),\ \dfrac{\partial^2 \hat{p}}{\partial \hat{x}_2^2}(\hat{a}_i),\ i = 1,2,3\ ; \\[2ex] \quad -\dfrac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{b}_1)\ ;\ -\dfrac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{b}_2)\ ;\ \dfrac{\sqrt{2}}{2}\left(\dfrac{\partial \hat{p}}{\partial \hat{x}_1} + \dfrac{\partial \hat{p}}{\partial \hat{x}_2}\right)(\hat{b}_3)\ ;\ \hat{p}(\hat{d}_i),\ i = 1,\ldots,6\ ; \\[2ex] \quad -\dfrac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{d}_i),\ i = 1,2\ ;\ -\dfrac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{d}_i),\ i = 3,4\ ;\ \dfrac{\sqrt{2}}{2}\left(\dfrac{\partial \hat{p}}{\partial \hat{x}_1} + \dfrac{\partial \hat{p}}{\partial \hat{x}_2}\right)(\hat{d}_i),\ i = 5,6\ ; \\[2ex] \quad \hat{p}(\hat{e}_i),\ i = 1,2,3] \end{array}\right.$$

Figure 2.2.1 : Basic finite element for the construction of a curved finite element C^1-compatible with the ARGYRIS triangle ($F_K \in (P_3)^2$).

Then, the set $\Sigma_K(v)$ of degrees of freedom of function $v \in \mathcal{C}^2(K)$ is given, in its local version $[DLLC(v)]$, by

$$(2.2.2)\quad \left\{\begin{array}{l} [DLLC(v)]_{1\times 24} = [v(a_1)\ \ v(a_2)\ \ v(a_3)\ \ Dv(a_1)(a_3-a_1)\ \ (\bar{s}-\underline{s})Dv(a_1)\vec{\chi}'(\underline{s}) \\[1.5ex] \quad (\underline{s}-\bar{s})Dv(a_2)\vec{\chi}'(\bar{s})\ \ Dv(a_2)(a_3-a_2)\ \ Dv(a_3)(a_2-a_3)\ \ Dv(a_3)(a_1-a_3) \\[1.5ex] \quad D^2v(a_1)(a_3-a_1)^2\ \ (\bar{s}-\underline{s})^2 D^2v(a_1)(\vec{\chi}'(\underline{s}))^2\ \ (\underline{s}-\bar{s})^2 D^2v(a_2)(\vec{\chi}'(\bar{s}))^2 \\[1.5ex] \quad D^2v(a_2)(a_3-a_2)^2\ \ D^2v(a_3)(a_2-a_3)^2\ \ D^2v(a_3)(a_1-a_3)^2\ \ D^2v(a_1)(a_2-a_3)^2 \\[1.5ex] \quad D^2v(a_2)(a_3-a_1)^2\ \ (\bar{s}-\underline{s})^2 D^2v(a_3)((\vec{\chi}'(\underline{s}),\vec{\chi}'(\bar{s}))\ \ Dv(b_1)(a_1-c_1) \\[1.5ex] \quad Dv(b_2)(a_2-c_2)\ \ Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3-\hat{b}_3)\ \ v(e_1)\ \ v(e_2)\ \ v(e_3)] \end{array}\right.$$

Step 2 : Transition from $\Sigma_K(v)$ to $\hat{\Delta}_K(v)$: Starting from the set $\Sigma_K(v)$ of 24 elements we introduce the set $\hat{\Delta}_K(v)$ of 36 values that we need to attribute to the set $\hat{\Sigma}$ of degrees of freedom in order to obtain a suitable interpolated function $\hat{w} \in \hat{P}$. In this way, it is convenient to introduce the following partition of the set $\hat{\Sigma}$:

$$(2.2.3) \qquad \left\{ \begin{array}{l} \hat{\Sigma}(\hat{p}) = \hat{\Sigma}_1(\hat{p}) \cup \hat{\Sigma}_2(\hat{p}) \cup \hat{\Sigma}_3(\hat{p}), \text{ where} \\[2mm] \hat{\Sigma}_1(\hat{p}) = \{(D^\alpha \hat{p}(\hat{a}_i) \; ; \; |\alpha| = 0,1,2 \; ; \; i = 1,2,3) \; ; \; \hat{p}(\hat{e}_i), \; i = 1,2,3\} \\[2mm] \hat{\Sigma}_2(\hat{p}) = \left\{ \hat{p}(\hat{d}_i), \; i = 1,...,4 \; ; \; -\dfrac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{b}_1) \; ; \; -\dfrac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{d}_i), \; i = 1,2 \; ; \right. \\[2mm] \qquad\qquad \left. -\dfrac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{b}_2) \; ; \; -\dfrac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{d}_i), \; i = 3,4 \right\} \\[2mm] \hat{\Sigma}_3(\hat{p}) = \left\{ \hat{p}(\hat{d}_i), \; i = 5,6 \; ; \; \dfrac{\sqrt{2}}{2}\left(\dfrac{\partial \hat{p}}{\partial \hat{x}_1} + \dfrac{\partial \hat{p}}{\partial \hat{x}_2}\right)(\hat{b}_3) \; ; \right. \\[2mm] \qquad\qquad \left. \dfrac{\sqrt{2}}{2}\left(\dfrac{\partial \hat{p}}{\partial \hat{x}_1} + \dfrac{\partial \hat{p}}{\partial \hat{x}_2}\right)(\hat{d}_i), \; i = 5,6 \right\} \end{array} \right.$$

and the associate partition of $\hat{\Delta}_K(v)$:

$$(2.2.4) \qquad \hat{\Delta}_K(v) = \hat{\Delta}_{K1}(v) \cup \hat{\Delta}_{K2}(v) \cup \hat{\Delta}_{K3}(v).$$

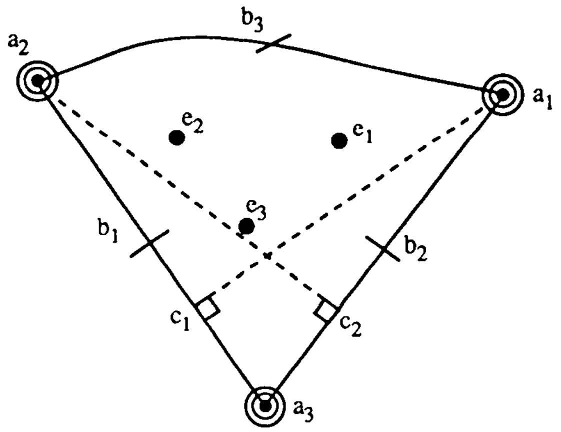


Figure 2.2.2 : The set of degrees of freedom Σ_K for the curved finite elements $\mathcal{C}^1$-compatible with the ARGYRIS finite elements ($F_K \in (P_3)^2$).

Then, from the knowledge of the set $\Sigma_K(v)$ of values of the degrees of freedom of v, we construct the corresponding set of values $\hat{\Delta}_{Ki}(v)$, $i = 1,2,3$:

Construction of $\hat{\Delta}_{K1}(v)$: The knowledge of the application F_K and of the set of values $\{(D^\alpha v(a_i),\; |\alpha| = 0,1,2 \; ; \; v(e_i)),\; i = 1,2,3\}$ leads immediately to $\hat{\Delta}_{K1}(v)$. It suffices to use relations (2.1.3) so that

$$(2.2.5) \qquad \hat{\Delta}_{K1}(v) = \{(D^\alpha \hat{v}(\hat{a}_i),\ |\alpha| = 0,1,2\ ;\ i = 1,2,3)\ ;\ \hat{v}(\hat{e}_i),\ i = 1,2,3\}.$$

This correspondence is detailed in section 3.1.

Construction of $\hat{\Delta}_{K2}(v)$: The construction of this set of values is much more tricky. Firstly, let us examine the case of the degrees of freedom of $\hat{\Sigma}_2$ located on the side $\hat{a}_3\hat{a}_1$. To verify conditions (2.1.2), the expected interpolating function $\pi_K v$ has to satisfy :

$*$ **On the one hand**, its trace $\pi_K v|_{[a_3,a_1]}$ coincides with the one-variable Hermite polynomial of degree 5, determined by the data of the degrees of freedom

$$(2.2.6) \qquad \begin{cases} \{v(a_1),\ v(a_3),\ Dv(a_1)(a_3-a_1),\ Dv(a_3)(a_1-a_3), \\ \qquad D^2v(a_1)(a_3-a_1)^2,\ D^2v(a_3)(a_1-a_3)^2\}. \end{cases}$$

We will use $\hat{x}_1$ as a parameterization variable of the side a_3a_1, i.e.,

$$(2.2.7) \qquad x_1 = x_{13} + (x_{11} - x_{13})\hat{x}_1, \quad x_2 = x_{23} + (x_{21} - x_{23})\hat{x}_1$$

where $x_{\alpha i}$ ($\alpha = 1,2$; $i = 1,2,3$) notes the α^{th}-coordinate of vertex a_i. Corresponding Hermite polynomial is named

$$(2.2.8) \qquad \hat{f}_1 \text{ (which has to coincide with } (\pi_K v)\circ F_K|_{[\hat{a}_3,\hat{a}_1]})$$

$*$ **On the other hand**, the trace $(D\pi_K v(.)(a_2-c_2))\circ F_K|_{\hat{a}_3,\hat{a}_1]}$ of its normal derivative $D\pi_K v(.)(a_2-c_2)$ coincides with the one variable Hermite polynomial of degree 4 determined by the data of the degrees of freedom

$$(2.2.9) \quad \{Dv(a_i)(a_2-c_2),\ D^2v(a_i)(a_2-c_2, a_1-a_3),\ i\in\{1,3\}\ ;\ Dv(b_2)(a_2-c_2)\}.$$

In a similar way to (2.2.7) and (2.2.8), the corresponding Hermite polynomial is named

$$(2.2.10) \qquad \hat{g}_1 \text{ (which has to coincide with } (D\pi_K v(.)(a_2-c_2))\circ F_K|_{[\hat{a}_3,\hat{a}_1]}).$$

By analogy with the above definitions we could define the following Hermite polynomials over the second straight side a_3a_2 of the triangle K, i.e.,

$$(2.2.11) \qquad x_1 = x_{13} + (x_{12} - x_{13})\hat{x}_2, \quad x_2 = x_{23} + (x_{22} - x_{23})\hat{x}_2$$

$$(2.2.12) \qquad \hat{f}_2 \text{ (which has to coincide with } (\pi_K v)\circ F_K|_{[\hat{a}_3,\hat{a}_2]})$$

$$(2.2.13) \qquad \hat{g}_2 \text{ (which has to coincide with } (D\pi_K v(.)(a_1-c_1))\circ F_K|_{[\hat{a}_3,\hat{a}_2]}).$$

Then, the set of values $\hat{\Delta}_{K2}(v)$ associated to the set of degrees of freedom $\hat{\Sigma}_2$ (see (2.1.5) (2.1.6) (2.2.3)) is given by

$$(2.2.14)\quad \begin{cases} \hat{\Delta}_{K2}(v) = [\hat{f}_2(\hat{d}_i),\ i=1,2\ ;\ \hat{f}_1(\hat{d}_i),\ i=3,4\ ; \\[2mm] -<\dfrac{\partial F_K}{\partial \hat{x}_1}(\hat{b}_1),\ \dfrac{a_2-a_3}{|a_2-a_3|^2}\,\dfrac{d\hat{f}_2}{d\hat{x}_2}(\hat{b}_1) + \dfrac{a_1-c_1}{|a_1-c_1|^2}\,\hat{g}_2(\hat{b}_1)>\ ; \\[2mm] -<\dfrac{\partial F_K}{\partial \hat{x}_1}(\hat{d}_i),\ \dfrac{a_2-a_3}{|a_2-a_3|^2}\,\dfrac{d\hat{f}_2}{d\hat{x}_2}(\hat{d}_i) + \dfrac{a_1-c_1}{|a_1-c_1|^2}\,\hat{g}_2(\hat{d}_i)>,\ i=1,2\ ; \\[2mm] -<\dfrac{\partial F_K}{\partial \hat{x}_2}(\hat{b}_2),\ \dfrac{a_1-a_3}{|a_1-a_3|^2}\,\dfrac{d\hat{f}_1}{d\hat{x}_1}(\hat{b}_2) + \dfrac{a_2-c_2}{|a_2-c_2|^2}\,\hat{g}_1(\hat{b}_2)>\ ; \\[2mm] -<\dfrac{\partial F_K}{\partial \hat{x}_2}(\hat{d}_i),\ \dfrac{a_1-a_3}{|a_1-a_3|^2}\,\dfrac{d\hat{f}_1}{d\hat{x}_1}(\hat{d}_i) + \dfrac{a_2-c_2}{|a_2-c_2|^2}\,\hat{g}_1(\hat{d}_i)>,\ i=3,4]. \end{cases}$$

Note that the definition of the last six elements uses relations (2.1.6) and (2.1.5), specifications (2.2.12) (2.2.13) (2.2.8) (2.2.10), and :

$$(2.2.15)\quad \begin{cases} \dfrac{d\hat{f}_2}{d\hat{x}_2} \text{ which will have to coincide with } (D\pi_K v(.)(a_2-a_3))\circ F_K|_{[\hat{a}_3,\hat{a}_2]} \\[2mm] \dfrac{d\hat{f}_1}{d\hat{x}_1} \text{ which will have to coincide with } (D\pi_K v(.)(a_1-a_3))\circ F_K|_{[\hat{a}_3,\hat{a}_1]} \end{cases}$$

Construction of $\hat{\Delta}_{K3}(v)$: It remains to compute the values that we have to attribute to the five degrees of freedom of $\hat{\Sigma}_3$. Firstly the correspondence $\hat{v} = v\circ F_K$ leads to

$$(2.2.16)\qquad D\hat{v}(\hat{b}_3)(\hat{a}_3-\hat{b}_3) = Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3-\hat{b}_3),$$

which is known since this value is among the degrees of freedom of Σ_K (see (2.2.2)).

Next, the values $D^\alpha\hat{v}(\hat{a}_i)$, $|\alpha| = 0,1,2$; $i = 1,2,3$ are listed in (2.2.5) while the side $\hat{a}_1\hat{a}_2$ of the triangle $\hat{K}$ can be parameterized by

$$(2.2.17)\qquad \hat{x}_1 = \hat{x}_1,\quad \hat{x}_2 = 1-\hat{x}_1.$$

Then, let $\hat{f}_3(\hat{x}_1)$ be the Hermite polynomial of degree 5, defined by the data of the degrees of freedom

$$(2.2.18)\quad \begin{cases} \{\hat{v}(\hat{a}_1),\ \hat{v}(\hat{a}_2),\ D\hat{v}(\hat{a}_1)(\hat{a}_2-\hat{a}_1),\ D\hat{v}(\hat{a}_2)(\hat{a}_1-\hat{a}_2), \\[2mm] \qquad D^2\hat{v}(\hat{a}_1)(\hat{a}_2-\hat{a}_1)^2,\ D^2\hat{v}(\hat{a}_2)(\hat{a}_1-\hat{a}_2)^2\} \end{cases}$$

and let $\hat{g}_3(\hat{x}_1)$ be the Hermite polynomial of degree 4, determined by the data of the degrees of freedom

$$(2.2.19)\quad \begin{cases} \{D\hat{v}(\hat{a}_\alpha)(\hat{a}_3-\hat{b}_3),\ \alpha=1,2\ ;\ D^2\hat{v}(\hat{a}_1)(\hat{a}_3-\hat{b}_3,\hat{a}_2-\hat{a}_1), \\[2mm] \qquad D^2\hat{v}(\hat{a}_2)(\hat{a}_3-\hat{b}_3,\hat{a}_1-\hat{a}_2),\ D\hat{v}(\hat{b}_3)(\hat{a}_3-\hat{b}_3)\}. \end{cases}$$

Then, we set

$$\hat{\Delta}_{K3}(v) = \left\{ \hat{f}_3\left(\frac{3}{4}\right) ; \hat{f}_3\left(\frac{1}{4}\right) ; -\sqrt{2}\hat{g}_3\left(\frac{1}{2}\right) ; -\sqrt{2}\hat{g}_3\left(\frac{3}{4}\right) ; -\sqrt{2}\hat{g}_3\left(\frac{1}{4}\right) \right\}. \tag{2.2.20}$$

Thus, only from the knowledge of the values $\Sigma_K(v)$ of the degrees of freedom of the function v, the relations (2.2.5) (2.2.14) and (2.2.20) assign one and only one value to every degree of freedom of $\hat{\Sigma}$. In view of the implementation, we will give a matrix presentation of these correspondences in paragraph 3.

Step 3 : Transition from the set of values $\hat{\Delta}_K(v)$ to the interpolate function $w = \pi_K v$: Let $\hat{w}$ be the function of $\hat{P}$ which takes the set of values $\hat{\Delta}_K(v)$ on the set of degrees of freedom $\hat{\Sigma}$ (see Figure 2.2.1). Then, to $\hat{w}$ defined on the reference triangle $\hat{K}$, relations (2.1.3) associate the function

$$\pi_K v = w = \hat{w} \circ (F_K)^{-1} \tag{2.2.21}$$

where the function $(F_K)^{-1}$ is the inverse of function $F_K : \hat{K} \longrightarrow K$ explicited in Example 1.3.1 (see relation (1.3.2)). ■

Finally, according to [2], this function w is exactly the expected function $\pi_K v$, i.e., $w = \pi_K v$, since :

Theorem 2.2.1 : The function w, defined in (2.2.21) is determined in a unique way by the data (2.2.2) of the local version $[DLLC(v)]$ of the set $\Sigma_K(v)$ of the values of the degrees of freedom of the function v to interpolate. Moreover the function w satisfies

$$\begin{cases} D^\alpha w(a_i) = D^\alpha v(a_i), \quad |\alpha| = 0,1,2 \; ; \; i = 1,2,3 \; ; \\ Dw(b_1)(a_1 - c_1) = Dv(b_1)(a_1 - c_1) \\ Dw(b_2)(a_2 - c_2) = Dv(b_2)(a_2 - c_2) \\ Dw(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) = Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) \\ w(e_i) = v(e_i), \quad i = 1,2,3 \end{cases} \tag{2.2.22}$$

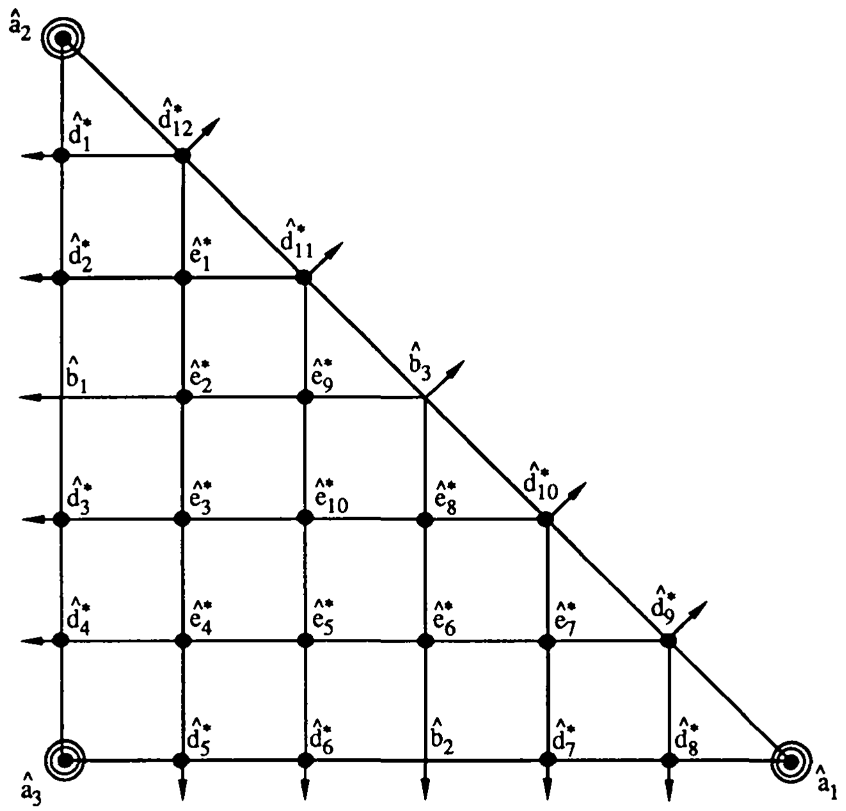
and the conditions (2.1.1) (2.1.2) and (2.1.4). Thus $w = \pi_K v$. ■

2.3 Definition of curved finite elements C^1-compatible with ARGYRIS elements for $F_K^* \in (P_5)^2$

Since the construction is very similar to the case $F_K \in (P_3)^2$ we only detail the new aspects. Firstly, relation (2.1.7) involves $\hat{P}_K^* \subset P_9$ (let us record that a star (*) makes the distinction between the cases $F_K \in (P_3)^2$ and $F_K^* \in (P_5)^2$).

2.3.1 The basic finite element $(\hat{K}, \hat{P}^*, \hat{\Sigma}^*)$

This element is described in Figure 2.3.1 and basis functions are given by (3.2.25).



$\hat{K}$ = unit right-angled triangle ; $\hat{P}^* = P_9$; $dim \hat{P}^* = 55$

$$
\left\{
\begin{array}{l}
\hat{\Sigma}^*(\hat{p}) = \Big\{ \hat{p}(\hat{a}_i),\ \dfrac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{a}_i),\ \dfrac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{a}_i),\ \dfrac{\partial^2 \hat{p}}{\partial \hat{x}_1^2}(\hat{a}_i),\ \dfrac{\partial^2 \hat{p}}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_i),\ \dfrac{\partial^2 \hat{p}}{\partial \hat{x}_2^2}(\hat{a}_i),\ i = 1,2,3\ ; \\[2ex]
\quad - \dfrac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{b}_1)\ ;\ - \dfrac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{b}_2)\ ;\ \dfrac{\sqrt{2}}{2}\left(\dfrac{\partial \hat{p}}{\partial \hat{x}_1} + \dfrac{\partial \hat{p}}{\partial \hat{x}_2}\right)(\hat{b}_3)\ ;\ \hat{p}(\hat{d}_i^*),\ i = 1,...,12\ ; \\[2ex]
\quad - \dfrac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{d}_i^*),\ i = 1,...,4\ ;\ - \dfrac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{d}_i^*),\ i = 5,...,8\ ; \\[2ex]
\quad \dfrac{\sqrt{2}}{2}\left(\dfrac{\partial \hat{p}}{\partial \hat{x}_1} + \dfrac{\partial \hat{p}}{\partial \hat{x}_2}\right)(\hat{d}_i^*),\ i = 9,...,12\ ;\ \hat{p}(\hat{e}_i^*),\ i = 1,...,10 \Big\}
\end{array}
\right.
$$

Figure 2.3.1 : Basic finite element for the construction of a curved finite element C^1-compatible with the ARGYRIS triangle $(F_K^* \in (P_5)^2)$.

2.3.2 Construction of the interpolating function $v \longrightarrow \pi_K^* v$

Like in section 2.2, this construction can be splitted in three steps :

Step 1 : Definition of the set Σ_K^* of degrees of freedom of the curved element. Set (see Figures 2.3.1 and 2.3.2) :

$$a_i = F_K^*(\hat{a}_i)\ ;\ \ b_i = F_K^*(\hat{b}_i),\ \ i = 1,2,3\ ;\ e_i^* = F_K^*(\hat{e}_i^*),\ \ i = 1,...,10 \tag{2.3.1}$$

while c_α is still the orthogonal projection of a_α on the side $a_3a_\alpha, \alpha = 1,2$. Then the set $\Sigma_K^*(v)$ of degrees of freedom of the function $v \in \mathcal{C}^2(K)$ is given, in its local version $[DLLC^*(v)]$, by

$$\left\{\begin{array}{l}\Sigma_K^*(v) = [DLLC^*(v)]_{1\times 31} = [v(a_1)\ \ v(a_2)\ \ v(a_3)\ \ Dv(a_1)(a_3-a_1)\\ (\bar{s}-\underline{s})Dv(a_1)\vec{\chi}'(\underline{s})\ \ (\underline{s}-\bar{s})Dv(a_2)\vec{\chi}'(\bar{s})\ \ Dv(a_2)(a_3-a_2)\\ Dv(a_3)(a_2-a_3)\ \ Dv(a_3)(a_1-a_3)\ \ D^2v(a_1)(a_3-a_1)^2\\ (\bar{s}-\underline{s})^2D^2v(a_1)(\vec{\chi}'(\underline{s}))^2\ \ (\underline{s}-\bar{s})^2D^2v(a_2)(\vec{\chi}'(\bar{s}))^2\ \ D^2v(a_2)(a_3-a_2)^2\\ D^2v(a_3)(a_2-a_3)^2\ \ D^2v(a_3)(a_1-a_3)^2\ \ D^2v(a_1)(a_2-a_3)^2\\ D^2v(a_2)(a_3-a_1)^2\ \ (\bar{s}-\underline{s})^2D^2v(a_3)((\vec{\chi}'(\underline{s}),\vec{\chi}'(\bar{s}))\ \ Dv(b_1)(a_1-c_1)\\ Dv(b_2)(a_2-c_2)\ \ Dv(b_3)DF_K^*(\hat{b}_3)(\hat{a}_3-\hat{b}_3)\ \ v(e_1^*)\ ...\ v(e_{10}^*)]\end{array}\right. \tag{2.3.2}$$

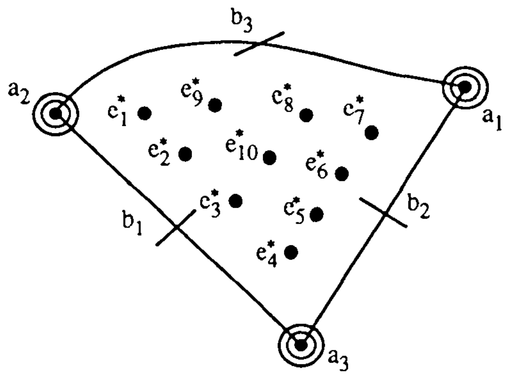


Figure 2.3.2 : The set of degrees of freedom Σ_K for the curved finite elements $\mathcal{C}^1$-compatible with the ARGYRIS finite element ($F_K^* \in (P_5)^2$).

Step 2 : Transition from $\Sigma_K^*(v)$ to $\hat{\Delta}_K^*(v)$: Starting from the set $\Sigma_K^*(v)$ including 31 elements we define the set $\hat{\Delta}_K^*(v)$ of 55 values that we need to attribute to the set $\hat{\Sigma}^*$ of degrees of freedom in order to obtain a suitable interpolated function $\hat{w} \in \hat{P}$. In this way we consider the following partition of the set $\hat{\Sigma}^*$:

$$
(2.3.3)\quad \begin{cases}
\hat\Sigma^*(\hat p) = \hat\Sigma_1^*(\hat p) \cup \hat\Sigma_2^*(\hat p) \cup \hat\Sigma_3^*(\hat p), \quad \text{where} \\[2mm]
\hat\Sigma_1^*(\hat p) = \{(D^\alpha \hat p(\hat a_i)\ ;\ |\alpha| = 0,1,2\ ;\ i=1,2,3)\ ;\ \hat p(\hat e_i^*),\ i=1,...,10\} \\[2mm]
\hat\Sigma_2^*(\hat p) = \Big\{\hat p(\hat d_i^*),\ i=1,...,8\ ;\ -\dfrac{\partial \hat p}{\partial \hat x_1}(\hat b_1)\ ;\ -\dfrac{\partial \hat p}{\partial \hat x_1}(\hat d_i),\ i=1,...,4\ ; \\[2mm]
\qquad -\dfrac{\partial \hat p}{\partial \hat x_2}(\hat b_2)\ ;\ -\dfrac{\partial \hat p}{\partial \hat x_2}(\hat d_i),\ i=5,...,8\Big\} \\[2mm]
\hat\Sigma_3^*(\hat p) = \Big\{\hat p(\hat d_i^*),\ i=9,...,12\ ;\ \dfrac{\sqrt2}{2}\left(\dfrac{\partial \hat p}{\partial \hat x_1} + \dfrac{\partial \hat p}{\partial \hat x_2}\right)(\hat b_3)\ ; \\[2mm]
\qquad \dfrac{\sqrt2}{2}\left(\dfrac{\partial \hat p}{\partial \hat x_1} + \dfrac{\partial \hat p}{\partial \hat x_2}\right)(\hat d_i^*),\ i=9,...,12\Big\}
\end{cases}
$$

Like in section 2.2, to $\Sigma_K^*(v)$ we associate the following partition of the set $\hat\Delta_K^*(v)$:

$$
(2.3.4)\qquad \hat\Delta_K^*(v) = \hat\Delta_{K1}^*(v) \cup \hat\Delta_{K2}^*(v) \cup \hat\Delta_{K3}^*(v)
$$

where each $\hat\Delta_{Ki}^*$ is associated to the corresponding $\hat\Sigma_i^*$, $i=1,2,3$. We get successively :

Construction of $\hat\Delta_{K1}^*(v)$: The knowledge of F_K^*, of the set of values $\{(D^\alpha v(a_i),\ |\alpha| = 0,1,2\ ;\ i=1,2,3)\ ;\ v(e_i^*),\ i=1,...,10\}$ and relation $\hat v = v \circ F_K^*$ immediately involve the knowledge of :

$$
(2.3.5)\qquad \hat\Delta_{K1}^*(v) = \{(D^\alpha \hat v(\hat a_i),\ |\alpha| = 0,1,2\ ;\ i=1,2,3)\ ;\ \hat v(\hat e_i^*),\ i=1,...,10\}.
$$

Construction of $\hat\Delta_{K2}^*(v)$: The functions $\hat f_1$, $\hat g_1$, $\hat f_2$, $\hat g_2$ determined in section 2.2 play exactly the same role here and, since $F_K|_{[\hat a_3,\hat a_1]} \equiv F_K^*|_{[\hat a_3,\hat a_1]}$ are affine, they are identical. Then, by similarity, we obtain

$$
(2.3.6)\quad \begin{cases}
\hat\Delta_{K2}^*(v) = [\hat f_2(\hat d_i^*),\ i=1,...,4\ ;\ \hat f_1(\hat d_i^*),\ i=5,...,8\ ; \\[2mm]
\quad -<\dfrac{\partial F_K^*}{\partial \hat x_1}(\hat b_1),\ \dfrac{a_2-a_3}{|a_2-a_3|^2}\dfrac{d\hat f_2}{d\hat x_2}(\hat b_1) + \dfrac{a_1-c_1}{|a_1-c_1|^2}\hat g_2(\hat b_1)>\ ; \\[2mm]
\quad -<\dfrac{\partial F_K^*}{\partial \hat x_1}(\hat d_i^*),\ \dfrac{a_2-a_3}{|a_2-a_3|^2}\dfrac{d\hat f_2}{d\hat x_2}(\hat d_i^*) + \dfrac{a_1-c_1}{|a_1-c_1|^2}\hat g_2(\hat d_i^*)>,\ i=1,...,4\ ; \\[2mm]
\quad -<\dfrac{\partial F_K^*}{\partial \hat x_2}(\hat b_2),\ \dfrac{a_1-a_3}{|a_1-a_3|^2}\dfrac{d\hat f_1}{d\hat x_1}(\hat b_2) + \dfrac{a_2-c_2}{|a_2-c_2|^2}\hat g_1(\hat b_2)>\ ; \\[2mm]
\quad -<\dfrac{\partial F_K^*}{\partial \hat x_2}(\hat d_i^*).\ \dfrac{a_1-a_3}{|a_1-a_3|^2}\dfrac{d\hat f_1}{d\hat x_1}(\hat d_i^*) + \dfrac{a_2-c_2}{|a_2-c_2|^2}\hat g_1(\hat d_i^*)>,\ i=5,...,8].
\end{cases}
$$

Construction of $\hat{\Delta}^*_{K3}(v)$: Since the restrictions to $[\hat{a}_1, \hat{a}_2]$ of F^*_K and F_K are different, the associated functions $\hat{f}^*_3$ and $\hat{g}^*_3$ are generally different from $\hat{f}_3$ and $\hat{g}_3$. We obtain

(2.3.7) $$\hat{\Delta}^*_{K3}(v) = \{\hat{f}^*_3(\hat{d}^*_i),\ i = 9, ..., 12\ ;\ -\sqrt{2}\hat{g}^*_3(\hat{b}_3)\ ;\ -\sqrt{2}\hat{g}^*_3(\hat{d}^*_i),\ i = 9, ..., 12\}.$$

Step 3 : Transition from $\hat{\Delta}^*_K(v)$ to the function $w^* = \pi^*_K v$: Let $\hat{w}^* \in \hat{P}^*$ be the polynomial of degree 9 which takes the values $\hat{\Delta}^*_K(v)$ on the set of degrees of freedom $\hat{\Sigma}^*$. Then

(2.3.8) $$\pi^*_K v = w^* = \hat{w}^* \circ (F^*_K)^{-1}.$$

■

In [2], we have proved that this function $w^* = \pi^*_K v$ satisfy all the expected results :

Theorem 2.3.1 : The function w^*, defined by (2.3.8), is determined in a unique way by the data of the set $\Sigma^*_K(v)$ of the values of the degrees of freedom of the function v. Moreover the function w^* satisfies

(2.3.9) $$\begin{cases} D^\alpha w^*(a_i) = D^\alpha v(a_i), \quad |\alpha| = 0, 1, 2\ ;\ i = 1, 2, 3\ ; \\ Dw^*(b_1)(a_1 - c_1) = Dv(b_1)(a_1 - c_1) \\ Dw^*(b_2)(a_2 - c_2) = Dv(b_2)(a_2 - c_2) \\ Dw^*(b_3)DF^*_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) = Dv(b_3)DF^*_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) \\ w(e^*_i) = v(e^*_i), \quad i = 1, ..., 10 \end{cases}$$

and the conditions (2.1.1) (2.1.2) **and** (2.1.4). Thus $w^* = \pi^*_K v$. ■

3 IMPLEMENTATION OF CURVED FINITE ELEMENTS $\mathcal{C}^1$-COMPATIBLE WITH ARGYRIS TRIANGLE

In this paragraph we detail the matrix decompositions which allow to easily realize the interpolation modules related to both curved finite elements $\mathcal{C}^1$ compatible with ARGYRIS triangle.

3.1 Interpolation modules associated to curved finite element $\mathcal{C}^1$-compatible with ARGYRIS triangle when $F_K \in (P_3)^2$

Let v be a function of $\mathcal{C}^2(K)$. By using the construction detailed in section 2.2, we are able to define its interpolate $\pi_K v$, i.e.,

(3.1.1) $$\pi_K v(x) = w(x) = \hat{w}(\hat{x}).$$

Subsequently we will use this function $\hat{w}$ on $\hat{K}$ instead of w on K. This is usual in finite element implementation : in particular this allows to use numerical integration schemes on $\hat{K}$. Nevertheless, note that in [9] we took advantage of the linearity of F_K (we only considered straight triangles), of the properties of barycentric coordinates and of the eccentricity parameters to work directly on K.

Now it remains to transcribe in matrix expressions the relation (3.1.1). By similarity with section 2.2, we consider three steps :

Step 1 : $v \longrightarrow \Sigma_K(v) = [DLLC(v)]$: The application F_K is given in Example 1.3.1 while $\Sigma_K(v)$ is the following set of values (see (2.2.2)) :

$$(3.1.2) \quad \left\{ \begin{array}{l} [DLLC(v)]_{1\times 24} = [v(a_1) \;\; v(a_2) \;\; v(a_3) \;\; Dv(a_1)(a_3-a_1) \;\; (\bar{s}-\underline{s})Dv(a_1)\vec{\chi}'(\underline{s}) \\ \quad -(\bar{s}-\underline{s})Dv(a_2)\vec{\chi}'(\bar{s}) \;\; Dv(a_2)(a_3-a_2) \;\; Dv(a_3)(a_2-a_3) \;\; Dv(a_3)(a_1-a_3) \\ \quad D^2v(a_1)(a_3-a_1)^2 \;\; (\bar{s}-\underline{s})^2 D^2v(a_1)(\vec{\chi}'(\underline{s}))^2 \;\; (\bar{s}-\underline{s})^2 D^2v(a_2)(\vec{\chi}'(\bar{s}))^2 \\ \quad D^2v(a_2)(a_3-a_2)^2 \;\; D^2v(a_3)(a_2-a_3)^2 \;\; D^2v(a_3)(a_1-a_3)^2 \;\; D^2v(a_1)(a_2-a_3)^2 \\ \quad D^2v(a_2)(a_3-a_1)^2 \;\; (\bar{s}-\underline{s})^2 D^2v(a_3)((\vec{\chi}'(\underline{s}),\vec{\chi}'(\bar{s})) \;\; Dv(b_1)(a_1-c_1) \\ \quad Dv(b_2)(a_2-c_2) \;\; Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3-\hat{b}_3) \;\; v(e_1) \;\; v(e_2) \;\; v(e_3)] \end{array} \right.$$

When the side a_1a_2 is straight, note that the first twenty degrees of freedom become identical to the first twenty local degrees of freedom of the ARGYRIS element (see [9, (3.1.18)]) since

$$(3.1.3) \qquad \vec{\chi}'(s) = \frac{1}{\bar{s}-\underline{s}} \; \Sigma_{\alpha=1}^{2} \; (\chi_\alpha(\bar{s}) - \chi_\alpha(\underline{s}))\vec{e}_\alpha = \frac{1}{\bar{s}-\underline{s}} \; (a_2 - a_1).$$

To the set (3.1.2), we associate the set of global degrees of freedom :

$$(3.1.4) \quad \left\{ \begin{array}{l} [DLGL(v)]_{1\times 24} = [v(a_1) \;\; v(a_2) \;\; v(a_3) \;\; \frac{\partial v}{\partial x_1}(a_1) \;\; \frac{\partial v}{\partial x_2}(a_1) \;\; \frac{\partial v}{\partial x_1}(a_2) \\ \quad \frac{\partial v}{\partial x_2}(a_2) \;\; \frac{\partial v}{\partial x_1}(a_3) \;\; \frac{\partial v}{\partial x_2}(a_3) \;\; \frac{\partial^2 v}{\partial x_1^2}(a_1) \;\; \frac{\partial^2 v}{\partial x_1 \partial x_2}(a_1) \;\; \frac{\partial^2 v}{\partial x_2^2}(a_1) \\ \quad \frac{\partial^2 v}{\partial x_1^2}(a_2) \;\; \frac{\partial^2 v}{\partial x_1 \partial v_2}(a_2) \;\; \frac{\partial^2 v}{\partial x_2^2}(a_2) \;\; \frac{\partial^2 v}{\partial x_1^2}(a_3) \;\; \frac{\partial^2 v}{\partial x_1 \partial x_2}(a_3) \;\; \frac{\partial^2 v}{\partial x_2^2}(a_3) \\ \quad \frac{\partial v}{\partial \nu_1}(b_1) \;\; \frac{\partial v}{\partial \nu_2}(b_2) \;\; Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3-\hat{b}_3) \;\; v(e_1) \;\; v(e_2) \;\; v(e_3)] \end{array} \right.$$

Local and global degrees of freedom are linked by the relation

$$(3.1.5) \qquad [DLLC(v)]_{1\times 24} = [DLGL(v)]_{1\times 24}[\dot{D}]_{24\times 24}$$

with

$$
\grave{D} = \begin{bmatrix} I_3 & & & & & & & \\ & \grave{d}_1 & & & & & & \\ & & \grave{d}_2 & & & \bigcirc & & \\ & & & \grave{d}_3 & & & & \\ & & & & \grave{d}_4 & & & \\ & & \bigcirc & & & n_1 & & \\ & & & & & & n_2 & \\ & & & & & & & I_4 \end{bmatrix} \tag{3.1.6}
$$

where, by using relations (1.3.3) and by noting $a_i(x_{1i}, x_{2i})$, we have

$$
\tilde{d}_1 = \begin{bmatrix} X_{31} & \psi_1'(0) \\ Y_{31} & \psi_2'(0) \end{bmatrix} ; \ \tilde{d}_2 = \begin{bmatrix} -\psi_1'(1) & X_{32} \\ -\psi_2'(1) & Y_{32} \end{bmatrix} ; \ \tilde{d}_3 = \begin{bmatrix} X_{23} & X_{13} \\ Y_{23} & Y_{13} \end{bmatrix} \tag{3.1.7}
$$

$$
\tilde{d}_4 = \begin{bmatrix}
(X_{31})^2 & (\psi_1'(0))^2 & 0 & 0 & 0 & 0 & (X_{32})^2 & 0 & 0 \\
2X_{31}Y_{31} & 2\psi_1'(0)\psi_2'(0) & 0 & 0 & 0 & 0 & 2X_{32}Y_{32} & 0 & 0 \\
(Y_{31})^2 & (\psi_2'(0))^2 & 0 & 0 & 0 & 0 & (Y_{32})^2 & 0 & 0 \\
0 & 0 & (\psi_1'(1))^2 & (X_{32})^2 & 0 & 0 & 0 & (X_{31})^2 & 0 \\
0 & 0 & 2\psi_1'(1)\psi_2'(1) & 2X_{32}Y_{32} & 0 & 0 & 0 & 2X_{31}Y_{31} & 0 \\
0 & 0 & (\psi_2'(1))^2 & (Y_{32})^2 & 0 & 0 & 0 & (Y_{31})^2 & 0 \\
0 & 0 & 0 & 0 & (X_{32})^2 & (X_{31})^2 & 0 & 0 & \psi_1'(0)\psi_1'(1) \\
0 & 0 & 0 & 0 & 2X_{32}Y_{32} & 2X_{31}Y_{31} & 0 & 0 & \left\{ \begin{array}{l} \psi_1'(0)\psi_2'(1) \\ +\psi_2'(0)\psi_1'(1) \end{array} \right. \\
0 & 0 & 0 & 0 & (Y_{32})^2 & (Y_{31})^2 & 0 & 0 & \psi_2'(0)\psi_2'(1)
\end{bmatrix} \tag{3.1.8}
$$

with $X_{ij} = x_{1i} - x_{1j}$, $Y_{ij} = x_{2i} - x_{2j}$, $1 \leq i, j \leq 3$, and n_1 and n_2 as given in [9, (3.1.25)].

Step 2 : Transition from $\Sigma_K(v) = [DLLC(v)]$ to $\hat{\Delta}_K(v) = [DL(\hat{w})]$: On the fixed reference triangle $\hat{K}$, i.e., the unit right-angled triangle, we introduce only one set of degrees of freedom (see Figure 2.2.1) :

$$
(3.1.9)\left\{
\begin{aligned}
&[DL(\hat{w})]_{1\times 36} = [\hat{w}(\hat{a}_1)\ \ \hat{w}(\hat{a}_2)\ \ \hat{w}(\hat{a}_3)\ \ \frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_1)\ \ \frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_1)\ \ \frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_2)\ \ \frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_2)\\
&\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_3)\ \ \frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_3)\ \ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_1)\ \ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_1)\ \ \frac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_1)\ \ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_2)\\
&\frac{\partial^2 \hat{w}}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_2)\ \ \frac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_2)\ \ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_3)\ \ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_3)\ \ \frac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_3)\\
&-\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{b}_1)\ \ -\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{b}_2)\ \ \frac{\sqrt{2}}{2}\left(\frac{\partial \hat{w}}{\partial \hat{x}_1}+\frac{\partial \hat{w}}{\partial \hat{x}_2}\right)(\hat{b}_3)\ \ \hat{w}(\hat{d}_1)\ \ \hat{w}(\hat{d}_2)\ \ \hat{w}(\hat{d}_3)\ \ \hat{w}(\hat{d}_4)\\
&\hat{w}(\hat{d}_5)\ \ \hat{w}(\hat{d}_6)\ \ -\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{d}_1)\ \ -\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{d}_2)\ \ -\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{d}_3)\ \ -\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{d}_4)\\
&\frac{\sqrt{2}}{2}\left(\frac{\partial \hat{w}}{\partial \hat{x}_1}+\frac{\partial \hat{w}}{\partial \hat{x}_2}\right)(\hat{d}_5)\ \ \frac{\sqrt{2}}{2}\left(\frac{\partial \hat{w}}{\partial \hat{x}_1}+\frac{\partial \hat{w}}{\partial \hat{x}_2}\right)(\hat{d}_6)\ \ \hat{w}(\hat{e}_1)\ \ \hat{w}(\hat{e}_2)\ \ \hat{w}(\hat{e}_3)]
\end{aligned}
\right.
$$

Then, step 2 of section 2.2 leads to

$$(3.1.10)\qquad [DL(\hat{w})]_{1\times 36} = [DLLC(v)]_{1\times 24}\,[B]_{24\times 36}$$

where the matrix B is piecewise constructed hereunder. Then relations (3.1.5) and (3.1.10) give

$$(3.1.11)\qquad [DL(\hat{w})]_{1\times 36} = [DLGL(v)]_{1\times 24}\,[\tilde{D}]_{24\times 24}\,[B]_{24\times 36}.$$

By consideration of relations (3.1.9) and (3.1.10), it is natural to realize the following partition of matrix B

$$(3.1.12)\qquad [B]_{24\times 36} = [B_1\ \ B_2\ \ B_3\ \ B_4\ \ B_5\ \ B_6\ \ B_7]$$

where submatrices B_i, $i = 1, \ldots, 7$, have 24 lines and respectively 3, 6, 9, 3, 6, 6 and 3 columns, and are determined as follows :

Construction of submatrix B_1 : From (2.2.21) and (2.2.22), one gets $\hat{w}(\hat{a}_i) = v(a_i)$, $i = 1, 2, 3$ so that

$$(3.1.13)\qquad {}^tB_1 = [I_3\ ;\ 0_{3\times 21}].$$

Construction of submatrix B_2 : Since $\hat{w} = w \circ F_K$, we obtain

$$\frac{\partial \hat{w}}{\partial \hat{x}_\alpha}(\hat{x}) = D\hat{w}(\hat{x})\vec{\hat{e}}_\alpha = Dw(x)DF_K(\hat{x})\vec{\hat{e}}_\alpha = Dw(x)\,\frac{\partial F_K}{\partial \hat{x}_\alpha}(\hat{x})$$

so that relation (2.2.22) involves

$$(3.1.14) \qquad \frac{\partial \hat{w}}{\partial \hat{x}_\alpha}(\hat{a}_i) = Dv(a_i)\,\frac{\partial F_K}{\partial \hat{x}_\alpha}(\hat{a}_i),\ i = 1,2,3$$

and with (1.3.2)

$$(3.1.15) \quad \begin{cases} \dfrac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_1) = Dv(a_1)(a_1 - a_3)\,; \ \dfrac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_1) = Dv(a_1)[a_1 - a_3 + (\bar{s} - \underline{s})\vec{\chi}'(\underline{s})] \\ \dfrac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_2) = Dv(a_2)[a_2 - a_3 - (\bar{s} - \underline{s})\vec{\chi}'(\bar{s})]\,; \ \dfrac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_2) = Dv(a_2)(a_2 - a_3) \\ \dfrac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_3) = Dv(a_3)(a_1 - a_3)\,; \ \dfrac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_3) = Dv(a_3)(a_2 - a_3) \end{cases}$$

From these expressions and relations (3.1.2) (3.1.9) (3.1.10) and (3.1.12) we obtain

$$(3.1.16) \qquad {}^tB_2 = [0_{6\times 3}\ ;\ {}^t(b_2)_{6\times 6}\ ;\ 0_{6\times 15}]$$

with

$$(3.1.17) \qquad b_2 = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Construction of submatrix B_3 : Relations $\hat{w} = w \circ F_K$ and (2.2.22) involve for $\alpha = 1,2$:

$$(3.1.18) \qquad \frac{\partial^2 \hat{w}}{\partial \hat{x}_\alpha \partial \hat{x}_\beta}(\hat{x}) = Dw(x)\,\frac{\partial^2 F_K}{\partial \hat{x}_\alpha \partial \hat{x}_\beta}(\hat{x}) + D^2 w(x)\left(\frac{\partial F_K}{\partial \hat{x}_\alpha}(\hat{x}),\ \frac{\partial F_K}{\partial \hat{x}_\beta}(\hat{x})\right)$$

so that with (1.3.2) we get

$$
(3.1.19)\quad \begin{cases}
\dfrac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_1) & = D^2 v(a_1)(a_1 - a_3)^2 \\
\dfrac{\partial^2 \hat{w}}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_1) & = D^2 v(a_1)(a_1 - a_3,\ a_1 - a_3 + (\bar{s} - \underline{s})\vec{\chi}'(\underline{s}))] \\
& \quad + Dv(a_1)[2(a_1 - a_2) + \dfrac{1}{2}(\bar{s} - \underline{s})(3\vec{\chi}'(\underline{s}) + \vec{\chi}'(\bar{s}))] \\
\dfrac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_1) & = D^2 v(a_1)[a_1 - a_3 + (\bar{s} - \underline{s})\vec{\chi}'(\underline{s})]^2 \\
& \quad - Dv(a_1)[2(a_1 - a_2) + (\bar{s} - \underline{s})(\vec{\chi}'(\underline{s}) + \vec{\chi}'(\bar{s}))] \\
\dfrac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_2) & = D^2 v(a_2)[a_2 - a_3 - (\bar{s} - \underline{s})\vec{\chi}'(\bar{s})]^2 \\
& \quad - Dv(a_2)[2(a_2 - a_1) - (\bar{s} - \underline{s})(\vec{\chi}'(\underline{s}) + \vec{\chi}'(\bar{s}))] \\
\dfrac{\partial^2 \hat{w}}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_2) & = D^2 v(a_2)[a_2 - a_3 - (\bar{s} - \underline{s})\vec{\chi}'(\bar{s}), a_2 - a_3] \\
& \quad + Dv(a_2)[2(a_2 - a_1) - \dfrac{1}{2}(\bar{s} - \underline{s})(3\vec{\chi}'(\bar{s}) + \vec{\chi}'(\underline{s}))] \\
\dfrac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_2) & = D^2 v(a_2)[a_2 - a_3]^2 \\
\dfrac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_3) & = D^2 v(a_3)[a_1 - a_3]^2 \\
\dfrac{\partial^2 \hat{w}}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_3) & = D^2 v(a_3)[a_1 - a_3,\ a_2 - a_3] - \dfrac{1}{2}(\bar{s} - \underline{s}) Dv(a_3)[\vec{\chi}'(\bar{s}) - \vec{\chi}'(\underline{s})] \\
\dfrac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_3) & = D^2 v(a_3)[a_2 - a_3]^2
\end{cases}
$$

From these expressions and relations (3.1.2) (3.1.9) (3.1.10) and (3.1.12), we deduce

$$
(3.1.20)\qquad {}^t B_3 = [0_{9\times 3}\ ;\ {}^t(b_{31})_{9\times 6}\ ;\ {}^t(b_{32})_{9\times 9}\ ;\ 0_{9\times 6}]
$$

where matrices b_{31} and b_{32} are given by relations (3.1.21) (3.1.22).

$$
(3.1.21)\quad b_{31} = \begin{bmatrix}
0 & \left(2\tilde{a}^1 + \frac{1}{2}\bar{\tilde{a}}^1\right) & -(2\tilde{a}^1 + \bar{\tilde{a}}^1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \left(\frac{3}{2} + 2\tilde{a}^2 + \frac{1}{2}\bar{\tilde{a}}^2\right) & -(1 + 2\tilde{a}^2 + \bar{\tilde{a}}^2) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -(1 + 2\tilde{b}^1 - \bar{\tilde{b}}^1) & \left(\frac{3}{2} + 2\tilde{b}^1 - \frac{1}{2}\bar{\tilde{b}}^1\right) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -(2\tilde{b}^2 - \bar{\tilde{b}}^2) & \left(2\tilde{b}^2 - \frac{1}{2}\bar{\tilde{b}}^2\right) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\tilde{c}^1 + \bar{\tilde{c}}^1) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\tilde{c}^2 + \bar{\tilde{c}}^2) & 0
\end{bmatrix}
$$

$$
(3.1.22) \quad b_{32} = \begin{bmatrix}
1 & \left(1+\frac{1+\tilde a^1}{2\tilde a^2}\right) & \left(1+\frac{1+\tilde a^1}{\tilde a^2}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\tilde a^2}{2(1+\tilde a^1)} & \left(1+\frac{\tilde a^2}{1+\tilde a^1}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \left(1+\frac{\tilde b^1}{1+\tilde b^2}\right) & \frac{\tilde b^1}{2(1+\tilde b^2)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \left(1+\frac{1+\tilde b^2}{\tilde b^1}\right) & \left(1+\frac{1+\tilde b^2}{2\tilde b^1}\right) & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\tilde c^1 \tilde{\tilde c}^1}{\tilde c^1 \tilde{\tilde c}^2 + \tilde c^2 \tilde{\tilde c}^1} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{\tilde c^2 \tilde{\tilde c}^2}{\tilde c^1 \tilde{\tilde c}^2 + \tilde c^2 \tilde{\tilde c}^1} & 0 \\
0 & \frac{-1}{2\tilde a^2(1+\tilde a^1)} & \frac{-1}{\tilde a^2(1+\tilde a^1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-1}{\tilde b^1(1+\tilde b^2)} & \frac{-1}{2\tilde b^1(1+\tilde b^2)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{\tilde c^1 \tilde{\tilde c}^2 + \tilde c^2 \tilde{\tilde c}^1} & 0
\end{bmatrix}
$$

It remains to indicate how to obtain the coefficients of the submatrices b_{31} and b_{32}. The second and third columns of b_{31} include terms in $Dv(a_1)$ that we have to express by using the fourth and fifth degrees of freedom of the matrix (3.1.2). Set

$$\vec A_1 = a_3 - a_1 \quad \text{and} \quad \vec A_2 = (\bar s - \underline s)\vec\chi'(\underline s)$$

and assume that these vectors are linearly independent. Then $\tilde a^\alpha$ and $\tilde{\tilde a}^\alpha$ are defined by

$$a_1 - a_2 = \tilde a^\alpha \vec A_\alpha \ ; \ (\bar s - \underline s)\vec\chi'(\bar s) = \tilde{\tilde a}^\alpha \vec A_\alpha$$

so that, if we denote $\vec e_3 = \vec e_1 \times \vec e_2$:

$$\begin{cases}
\tilde a^1 = \dfrac{[(a_1 - a_2) \times \vec A_2].\vec e_3}{(\vec A_1 \times \vec A_2).\vec e_3} & ; \ \tilde a^2 = \dfrac{[\vec A_1 \times (a_1 - a_2)].\vec e_3}{(\vec A_1 \times \vec A_2).\vec e_3} \\[2ex]
\tilde{\tilde a}^1 = (\bar s - \underline s)\, \dfrac{[\vec\chi'(\bar s) \times \vec A_2].\vec e_3}{(\vec A_1 \times \vec A_2).\vec e_3} & ; \ \tilde{\tilde a}^2 = (\bar s - \underline s)\, \dfrac{[\vec A_1 \times \vec\chi'(\bar s)].\vec e_3}{(\vec A_1 \times \vec A_2).\vec e_3}.
\end{cases}$$

Similarly, set

$$\vec B_1 = -(\bar s - \underline s)\vec\chi'(\bar s) \ ; \ \vec B_2 = a_3 - a_2 \ ; \ a_2 - a_1 = \tilde b^\alpha \vec B_\alpha \ ; \ (\bar s - \underline s)\vec\chi'(\underline s) = \tilde{\tilde b}^\alpha \vec B_\alpha$$
$$\vec C_1 = a_2 - a_3 \ ; \ \vec C_2 = a_1 - a_3 \ ; \ (\bar s - \underline s)\vec\chi'(\underline s) = \tilde c^\alpha \vec C_\alpha \ ; \ -(\bar s - \underline s)\vec\chi'(\bar s) = \tilde{\tilde c}^\alpha \vec C_\alpha.$$

From these definitions, we finally obtain :

$$(3.1.23)\quad \begin{cases} \tilde a^1 = \dfrac{(\vec B_2\times\vec A_2).\vec e_3}{(\vec A_1\times\vec A_2).\vec e_3} - 1 & ;\ \tilde a^2 = \dfrac{(\vec A_1\times\vec B_2).\vec e_3}{(\vec A_1\times\vec A_2).\vec e_3} \\[2ex] \tilde{\tilde a}^1 = -\dfrac{(\vec B_1\times\vec A_2).\vec e_3}{(\vec A_1\times\vec A_2).\vec e_3} & ;\ \tilde{\tilde a}^2 = -\dfrac{(\vec A_1\times\vec B_1).\vec e_3}{(\vec A_1\times\vec A_2).\vec e_3} \\[2ex] \tilde b^1 = \dfrac{(\vec A_1\times\vec B_2).\vec e_3}{(\vec B_1\times\vec B_2).\vec e_3} & ;\ \tilde b^2 = \dfrac{(\vec B_1\times\vec A_1).\vec e_3}{(\vec B_1\times\vec B_2).\vec e_3} - 1 \\[2ex] \tilde{\tilde b}^1 = \dfrac{(\vec A_2\times\vec B_2).\vec e_3}{(\vec B_1\times\vec B_2).\vec e_3} & ;\ \tilde{\tilde b}^2 = \dfrac{(\vec B_1\times\vec A_2).\vec e_3}{(\vec B_1\times\vec B_2).\vec e_3} \\[2ex] \tilde c^1 = -\dfrac{(\vec A_1\times\vec A_2).\vec e_3}{(\vec A_1\times\vec B_2).\vec e_3} = -\dfrac{1}{\tilde a^2} & ;\ \tilde c^2 = -\dfrac{(\vec A_2\times\vec B_2).\vec e_3}{(\vec A_1\times\vec B_2).\vec e_3} = -\dfrac{\tilde{\tilde b}^1}{\tilde b^1} \\[2ex] \tilde{\tilde c}^1 = -\dfrac{(\vec A_1\times\vec B_1).\vec e_3}{(\vec A_1\times\vec B_2).\vec e_3} = \dfrac{\tilde{\tilde a}^2}{\tilde a^2} & ;\ \tilde{\tilde c}^2 = -\dfrac{(\vec B_1\times\vec B_2).\vec e_3}{(\vec A_1\times\vec B_2).\vec e_3} = -\dfrac{1}{\tilde b^1} \end{cases}$$

Construction of submatrices B_i, $i = 4,5,6$: We have seen in section 2.2 that it is convenient to introduce the functions $\hat f_1$ (see (2.2.8)), $\hat f_2$ (see (2.2.12)), $\hat f_3$ (see (2.2.18)), $\hat g_1$ (see (2.2.10)), $\hat g_2$ (see (2.2.13)) and $\hat g_3$ (see (2.2.19)). Relations (1.3.2) (3.1.2), $\hat v = v\circ F_K$ and some technical computations prove that these functions $\hat f_i$, $\hat g_i$, $i = 1,2,3$ can be written as

$$(3.1.24)\quad \begin{cases} [\hat f_1(\hat x_1)\ \ \hat f_2(\hat x_2)\ \ \hat f_3(\hat x_1)] = [DLLC(v)]_{1\times 24}\ \ [\hat{\mathcal F}_1(\hat x_1)\ \ \hat{\mathcal F}_2(\hat x_2)\ \ \hat{\mathcal F}_3(\hat x_1)]_{24\times 3} \\ [\hat g_1(\hat x_1)\ \ \hat g_2(\hat x_2)\ \ \hat g_3(\hat x_1)] = [DLLC(v)]_{1\times 24}\ \ [\hat{\mathcal G}_1(\hat x_1)\ \ \hat{\mathcal G}_2(\hat x_2)\ \ \hat{\mathcal G}_3(\hat x_1)]_{24\times 3} \end{cases}$$

with

$$(3.1.25)\quad \begin{cases} {}^t[\hat{\mathcal F}_1(\hat x_1)] = [\hat x_1^3(6\hat x_1^2 - 15\hat x_1 + 10)\ ;\ 0\ ;\ (1-\hat x_1)^3(6\hat x_1^2 + 3\hat x_1 + 1)\ ; \\[1ex] \quad \hat x_1^3(1-\hat x_1)(4-3\hat x_1)\ ;\ 0\,0\,0\,0\ ;\ \hat x_1(1-\hat x_1)^3(1+3\hat x_1)\ ; \\[1ex] \quad \dfrac{1}{2}\,\hat x_1^3(1-\hat x_1)^2\ ;\ 0\,0\,0\,0\ ;\ \dfrac{1}{2}\,\hat x_1^2(1-\hat x_1)^3\ ;\ 0\,0\,0\,0\,0\,0\,0\,0\,0] \end{cases}$$

$$(3.1.26)\quad \begin{cases} {}^t[\hat{\mathcal F}_2(\hat x_2)] = [0\ ;\ \hat x_2^3(6\hat x_2^2 - 15\hat x_2 + 10)\ ;\ (1-\hat x_2)^3(6\hat x_2^2 + 3\hat x_2 + 1)\ ; \\[1ex] \quad 0\,0\,0\ ;\ \hat x_2^3(1-\hat x_2)(4-3\hat x_2)\ ;\ \hat x_2(1-\hat x_2)^3(1+3\hat x_2)\ ;\ 0\,0\,0\,0\ ; \\[1ex] \quad \dfrac{1}{2}\,\hat x_2^3(1-\hat x_2)^2\ ;\ \dfrac{1}{2}\,\hat x_2^2(1-\hat x_2)^3\ ;\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0] \end{cases}$$

$$
(3.1.27)\quad \begin{cases}
{}^t[\hat{\mathcal{F}}_3(\hat{x}_1)] = [\hat{x}_1^3(6\hat{x}_1^2 - 15\hat{x}_1 + 10)\ ;\ (1-\hat{x}_1)^3(6\hat{x}_1^2 + 3\hat{x}_1 + 1)\ ;\ 0\ ; \\[2ex]
\quad - (3\tilde{a}^1 + \tilde{\tilde{a}}^1)\hat{x}_1^3(1-\hat{x}_1)^2\ ;\ \hat{x}_1^3(1-\hat{x}_1)\{2 - \hat{x}_1 - (1-\hat{x}_1)(3\tilde{a}^2 + \tilde{\tilde{a}}^2)\} \\[2ex]
\quad + \hat{x}_1(1-\hat{x}_1)^3\{1 + \hat{x}_1 - \hat{x}_1(3\tilde{b}^1 - \tilde{\tilde{b}}^1)\}\ ;\ - \hat{x}_1^2(1-\hat{x}_1)^3(3\tilde{b}^2 - \tilde{\tilde{b}}^2)\ ; \\[2ex]
\quad 0\ 0\ 0\ ;\ \dfrac{1}{2}\,\hat{x}_1^3(1-\hat{x}_1)^2\ ;\ \dfrac{1}{2}\,\hat{x}_1^2(1-\hat{x}_1)^3\ ;\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]
\end{cases}
$$

$$
(3.1.28)\quad \begin{cases}
{}^t[\hat{\mathcal{G}}_1(\hat{x}_1)] = [0\ 0\ 0\ ;\ -\dfrac{1}{2}(1-\eta_2+2\tilde{a}^1)\hat{x}_1^2(2\hat{x}_1-1)(5-4\hat{x}_1)\ ; \\[2ex]
\quad - \tilde{a}^2\hat{x}_1^2(2\hat{x}_1-1)(5-4\hat{x}_1)\ ;\ 0\ 0\ ;\ (1-\hat{x}_1)^2(1-2\hat{x}_1)(1+4\hat{x}_1)\ ; \\[2ex]
\quad - \dfrac{1}{2}(1+\eta_2)(1-\hat{x}_1)^2(1-2\hat{x}_1)(1+4\hat{x}_1)\ ;\ \dfrac{1}{2}(\tilde{a}^1-\eta_2)\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1)\ ; \\[2ex]
\quad - \dfrac{1}{2}\ \dfrac{(\tilde{a}^2)^2}{1+\tilde{a}^1}\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1)\ ;\ 0\ 0\ ;\ \dfrac{-\tilde{c}^1\tilde{\tilde{c}}^1}{\tilde{c}^1\tilde{\tilde{c}}^2+\tilde{c}^2\tilde{\tilde{c}}^1}\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1)\ ; \\[2ex]
\quad - \left\{\dfrac{1}{2}(1+\eta_2) + \dfrac{\tilde{c}^2\tilde{\tilde{c}}^2}{\tilde{c}^1\tilde{\tilde{c}}^2+\tilde{c}^2\tilde{\tilde{c}}^1}\right\}\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1)\ ;\ \dfrac{1}{2(1+\tilde{a}^1)}\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1)\ ; \\[2ex]
\quad 0\ ;\ \dfrac{-1}{\tilde{c}^1\tilde{\tilde{c}}^2+\tilde{c}^2\tilde{\tilde{c}}^1}\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1)\ ;\ 0\ ;\ 16\hat{x}_1^2(1-\hat{x}_1)^2\ ;\ 0\ 0\ 0\ 0]
\end{cases}
$$

$$
(3.1.29)\quad \begin{cases}
{}^t[\hat{\mathcal{G}}_2(\hat{x}_2)] = [0\ 0\ 0\ 0\ 0\ ;\ - \tilde{b}^1\hat{x}_2^2(2\hat{x}_2-1)(5-4\hat{x}_2)\ ; \\[2ex]
\quad - \dfrac{1}{2}(2\tilde{b}^2+1+\eta_1)\hat{x}_2^2(2\hat{x}_2-1)(5-4\hat{x}_2)\ ;\ - \dfrac{1}{2}(1-\eta_1)(1-\hat{x}_2)^2(1-2\hat{x}_2)(1+4\hat{x}_2)\ ; \\[2ex]
\quad (1-\hat{x}_2)^2(1-2\hat{x}_2)(1+4\hat{x}_2)\ ;\ 0\ 0\ ;\ - \dfrac{1}{2}\ \dfrac{(\tilde{b}^1)^2}{1+\tilde{b}^2}\hat{x}_2^2(1-\hat{x}_2)(1-2\hat{x}_2)\ ; \\[2ex]
\quad \dfrac{1}{2}(\tilde{b}^2+\eta_1)\hat{x}_2^2(1-\hat{x}_2)(1-2\hat{x}_2)\ ;\ - \dfrac{1}{2}\left\{\dfrac{2\tilde{c}^1\tilde{\tilde{c}}^1}{\tilde{c}^1\tilde{\tilde{c}}^2+\tilde{c}^2\tilde{\tilde{c}}^1} + 1 - \eta_1\right\}\hat{x}_2(1-\hat{x}_2)^2(1-2\hat{x}_2)\ ; \\[2ex]
\quad - \dfrac{\tilde{c}^2\tilde{\tilde{c}}^2}{\tilde{c}^1\tilde{\tilde{c}}^2+\tilde{c}^2\tilde{\tilde{c}}^1}\hat{x}_2(1-\hat{x}_2)^2(1-2\hat{x}_2)\ ;\ 0\ ;\ \dfrac{1}{2}\ \dfrac{1}{1+\tilde{b}^2}\ \hat{x}_2^2(1-\hat{x}_2)(1-2\hat{x}_2)\ ; \\[2ex]
\quad - \dfrac{1}{\tilde{c}^1\tilde{\tilde{c}}^2+\tilde{c}^2\tilde{\tilde{c}}^1}\hat{x}_2(1-\hat{x}_2)^2(1-2\hat{x}_2)\ ;\ 16(1-\hat{x}_2)^2\hat{x}_2^2\ ;\ 0\ 0\ 0\ 0\ 0]
\end{cases}
$$

$$
(3.1.30) \left\{
\begin{aligned}
&{}^t[\hat{\mathcal{G}}_3(\hat{x}_1)] = [0\ 0\ 0\ ;\ \hat{x}_1^2(2\hat{x}_1-1)\left\{5-4\hat{x}_1+(1-\hat{x}_1)\left(\tilde{a}^1+\frac{1}{2}\,\tilde{\tilde{a}}^1\right)\right\}\ ;\\
&\hat{x}_1^2(2\hat{x}_1-1)\left\{-2+\frac{3}{2}\,\hat{x}_1+(1-\hat{x}_1)\left(\tilde{a}^2+\frac{1}{2}\,\tilde{\tilde{a}}^2\right)\right\}\ ;\\
&-\frac{1}{2}\,(1-\hat{x}_1)^2(1-2\hat{x}_1)\left\{(1+3\hat{x}_1)-(2\tilde{b}^1-\tilde{\tilde{b}}^1)\hat{x}_1\right\}\ ;\\
&(1-\hat{x}_1)^2(1-2\hat{x}_1)\left\{1+4\hat{x}_1+\frac{1}{2}\,(2\tilde{b}^2-\tilde{\tilde{b}}^2)\hat{x}_1\right\}\ 0\ 0\ ;\\
&\frac{1+\tilde{a}^1}{2\tilde{a}^2}\,\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1)\ ;\ \frac{1}{2}\left(1+\frac{\tilde{a}^2}{1+\tilde{a}^1}\right)\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1)\ ;\\
&-\frac{1}{2}\left(1+\frac{\tilde{b}^1}{1+\tilde{b}^2}\right)\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1)\ ;\ -\frac{1+\tilde{b}^2}{2\tilde{b}^1}\,\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1)\ ;\\
&0\ 0\ ;\ -\frac{1}{2(1+\tilde{a}^1)\tilde{a}^2}\,\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1)\ ;\\
&\frac{1}{2\tilde{b}^1(1+\tilde{b}^2)}\,\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1)\ ;\ 0\ 0\ 0\ ;\ 16(1-\hat{x}_1)^2\hat{x}_1^2\ ;\ 0\ 0\ 0]
\end{aligned}
\right.
$$

To express $\hat{\mathcal{G}}_1(\hat{x}_1)$ and $\hat{\mathcal{G}}_2(\hat{x}_2)$, we have used the eccentricity parameters η_1 and η_2 of the triangle $a_1a_2a_3$ which are given by (see [9, p. 69]) :

$$
a_1 - c_1 = a_1 - a_2 - \frac{1}{2}\,(1+\eta_1)(a_3-a_2)\ ;\ a_2 - c_2 = a_2 - a_1 - \frac{1}{2}\,(1-\eta_2)(a_3-a_1).
$$

Then we are able to construct the submatrices B_i, $i = 4, 5, 6$.

Construction of submatrix B_4 : Relations (2.2.3) and (2.2.14) involve

$$
(3.1.31) \quad -\frac{\partial\hat{w}}{\partial\hat{x}_1}\,(\hat{b}_1) = -\ < \frac{\partial F_K}{\partial\hat{x}_1}\,(\hat{b}_1),\ \frac{a_2-a_3}{|a_2-a_3|^2}\,\frac{d\hat{f}_2}{d\hat{x}_2}\left(\frac{1}{2}\right) + \frac{a_1-c_1}{|a_1-c_1|^2}\hat{g}_2\left(\frac{1}{2}\right) >
$$

where $< . , . >$ notes the usual scalar product in $\mathbb{R}^2$. Hence relations (3.1.24) (3.1.26) (3.1.28) and (3.1.31) give

$$
(3.1.32) \qquad -\frac{\partial\hat{w}}{\partial\hat{x}_1}\,(\hat{b}_1) = [DLLC(v)]_{1\times 24}\left[E^1\,\frac{d}{d\hat{x}_2}\,\hat{\mathcal{F}}_2\left(\frac{1}{2}\right) + E^2\,\hat{\mathcal{G}}_2\left(\frac{1}{2}\right)\right]_{24\times 1}
$$

where we have set

$$
(3.1.33) \quad \begin{cases} E^1 =< a_3 - a_1 + \frac{1}{4}(a_1 - a_2) + \frac{1}{8}(\bar{s} - \underline{s})\{3\vec{\chi}'(\bar{s}) - \vec{\chi}'(\underline{s})\}, \frac{a_2 - a_3}{|a_2 - a_3|} > \\ \\ E^2 =< a_3 - a_1 + \frac{1}{4}(a_1 - a_2) + \frac{1}{8}(\bar{s} - \underline{s})\{3\vec{\chi}'(\bar{s}) - \vec{\chi}'(\underline{s})\}, \frac{a_1 - c_1}{|a_1 - c_1|^2} > \end{cases}
$$

since (1.3.2) involves

$$
-\frac{\partial F_K}{\partial \hat{x}_1}(\hat{b}_1) = a_3 - a_1 + \frac{1}{4}(a_1 - a_2) + \frac{1}{8}(\bar{s} - \underline{s})\{3\vec{\chi}'(\bar{s}) - \vec{\chi}'(\underline{s})\}
$$

By similarity, we obtain

$$
(3.1.34) \qquad -\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{b}_2) = [DLLC(v)]_{1\times 24}\left[F^1 \frac{d}{d\hat{x}_1}\hat{\mathcal{F}}_1\left(\frac{1}{2}\right) + F^2\hat{\mathcal{G}}_1\left(\frac{1}{2}\right)\right]_{24\times 1}
$$

where

$$
(3.1.35) \quad \begin{cases} F^1 =< a_3 - a_2 + \frac{1}{4}(a_2 - a_1) - \frac{1}{8}(\bar{s} - \underline{s})\{3\vec{\chi}'(\underline{s}) - \vec{\chi}'(\bar{s})\}, \frac{a_1 - a_3}{|a_1 - a_3|^2} > \\ \\ F^2 =< a_3 - a_2 + \frac{1}{4}(a_2 - a_1) - \frac{1}{8}(\bar{s} - \underline{s})\{3\vec{\chi}'(\underline{s}) - \vec{\chi}'(\bar{s})\}, \frac{a_2 - c_2}{|a_2 - c_2|^2} > \end{cases}
$$

Moreover

$$
(3.1.36) \qquad \begin{cases} \frac{\sqrt{2}}{2}\left(\frac{\partial \hat{w}}{\partial \hat{x}_1} + \frac{\partial \hat{w}}{\partial \hat{x}_2}\right)(\hat{b}_3) = -\sqrt{2}D\hat{w}(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) \\ \\ \qquad = [DLLC(v)]_{1\times 24}\, {}^t[0_{1\times 20}\ ;\ -\sqrt{2}\ ;\ 0\ 0\ 0]. \end{cases}
$$

Thus, relations (3.1.32) (3.1.34) and (3.1.36) lead to

$$
(3.1.37)\ B_4 = \left[E^1 \frac{d}{d\hat{x}_2}\hat{\mathcal{F}}_2\left(\frac{1}{2}\right) + E^2\,\hat{\mathcal{G}}_2\left(\frac{1}{2}\right)\ ;\ F^1 \frac{d}{d\hat{x}_1}\hat{\mathcal{F}}_1\left(\frac{1}{2}\right) + F^2\,\hat{\mathcal{G}}_1\left(\frac{1}{2}\right)\ ;\ \begin{pmatrix} 0_{20\times 1} \\ -\sqrt{2} \\ 0_{3\times 1} \end{pmatrix}\right].
$$

Construction of submatrix B_5 : Relations (2.2.3) (2.2.14) (2.2.20) and (3.1.24) give

$$
(3.1.38) \qquad B_5 = \left[\hat{\mathcal{F}}_2\left(\frac{3}{4}\right)\ ;\ \hat{\mathcal{F}}_2\left(\frac{1}{4}\right)\ ;\ \hat{\mathcal{F}}_1\left(\frac{1}{4}\right)\ ;\ \hat{\mathcal{F}}_1\left(\frac{3}{4}\right)\ ;\ \hat{\mathcal{F}}_3\left(\frac{3}{4}\right)\ ;\ \hat{\mathcal{F}}_3\left(\frac{1}{4}\right)\right]_{24\times 6}
$$

Construction of submatrix B_6 : By similarity with the obtention of (3.1.32) (3.1.34) and (3.1.36) we get

$$
(3.1.39)\quad \begin{cases} B_6 = \Big[G^1 \dfrac{d\hat{\mathcal{F}}_2}{d\hat{x}_2}\left(\dfrac{3}{4}\right) + G^2\hat{\mathcal{G}}_2\left(\dfrac{3}{4}\right) \ ; \ H^1\, \dfrac{d\hat{\mathcal{F}}_2}{d\hat{x}_2}\left(\dfrac{1}{4}\right) + H^2\hat{\mathcal{G}}_2\left(\dfrac{1}{4}\right) \ ; \\[2ex] \qquad J^1 \dfrac{d\hat{\mathcal{F}}_1}{d\hat{x}_1}\left(\dfrac{1}{4}\right) + J^2\hat{\mathcal{G}}_1\left(\dfrac{1}{4}\right) \ ; \ K^1\, \dfrac{d\hat{\mathcal{F}}_1}{d\hat{x}_1}\left(\dfrac{3}{4}\right) + K^2\hat{\mathcal{G}}_1\left(\dfrac{3}{4}\right) \ ; \\[2ex] \qquad -\sqrt{2}\,\hat{\mathcal{G}}_3\left(\dfrac{3}{4}\right) \ ; \ -\sqrt{2}\,\hat{\mathcal{G}}_3\left(\dfrac{1}{4}\right)\Big]_{24\times 6} \end{cases}
$$

where we have set

$$
(3.1.40)\quad \begin{cases} G^1 = -< a_1 - a_3 + \dfrac{9}{16}\,(a_2 - a_1) + \dfrac{3}{32}\,(\bar{s} - \underline{s})\{\vec{\chi}'(\underline{s}) - 7\vec{\chi}'(\bar{s})\},\ \dfrac{a_2 - a_3}{|a_2 - a_3|^2} > \\[2ex] G^2 = -< a_1 - a_3 + \dfrac{9}{16}\,(a_2 - a_1) + \dfrac{3}{32}\,(\bar{s} - \underline{s})\{\vec{\chi}'(\underline{s}) - 7\vec{\chi}'(\bar{s})\},\ \dfrac{a_1 - c_1}{|a_1 - c_1|^2} > \\[2ex] H^1 = -< a_1 - a_3 + \dfrac{1}{16}\,(a_2 - a_1) + \dfrac{1}{32}\,(\bar{s} - \underline{s})\{3\vec{\chi}'(\underline{s}) - 5\vec{\chi}'(\bar{s})\},\ \dfrac{a_2 - a_3}{|a_2 - a_3|^2} > \\[2ex] H^2 = -< a_1 - a_3 + \dfrac{1}{16}\,(a_2 - a_1) + \dfrac{1}{32}\,(\bar{s} - \underline{s})\{3\vec{\chi}'(\underline{s}) - 5\vec{\chi}'(\bar{s})\},\ \dfrac{a_1 - c_1}{|a_1 - c_1|^2} > \\[2ex] J^1 = -< a_2 - a_3 + \dfrac{1}{16}\,(a_1 - a_2) - \dfrac{1}{32}\,(\bar{s} - \underline{s})\{3\vec{\chi}'(\bar{s}) - 5\vec{\chi}'(\underline{s})\},\ \dfrac{a_1 - a_3}{|a_1 - a_3|^2} > \\[2ex] J^2 = -< a_2 - a_3 + \dfrac{1}{16}\,(a_1 - a_2) - \dfrac{1}{32}\,(\bar{s} - \underline{s})\{3\vec{\chi}'(\bar{s}) - 5\vec{\chi}'(\underline{s})\},\ \dfrac{a_2 - c_2}{|a_2 - c_2|^2} > \\[2ex] K^1 = -< a_2 - a_3 + \dfrac{9}{16}\,(a_1 - a_2) - \dfrac{3}{32}\,(\bar{s} - \underline{s})\{\vec{\chi}'(\bar{s}) - 7\vec{\chi}'(\underline{s})\},\ \dfrac{a_1 - a_3}{|a_1 - a_3|^2} > \\[2ex] K^2 = -< a_2 - a_3 + \dfrac{9}{16}\,(a_1 - a_2) - \dfrac{3}{32}\,(\bar{s} - \underline{s})\{\vec{\chi}'(\bar{s}) - 7\vec{\chi}'(\underline{s})\},\ \dfrac{a_2 - c_2}{|a_2 - c_2|^2} > \end{cases}
$$

Construction of submatrix B_7 : Since $\hat{w}(\hat{e}_i) = w(e_i) = v(e_i)$, $i = 1, 2, 3$, we get

$$
(3.1.41)\qquad {}^tB_7 = [0_{3\times 21}\ ;\ I_{3\times 3}]
$$

Step 3 : Transition from $\hat{\Delta}_K(v)$ to the function $\hat{w}(\hat{x}) = w(x) = \pi_K v(x)$: By using the finite element described in Figure 2.2.1, we construct the function $\hat{w} \in \hat{P} = P_7$ which takes the values $\hat{\Delta}_K(v)$ on the set of degrees of freedom $\hat{\Sigma}$ so that, with relation (3.1.9)

$$
(3.1.42)\qquad \hat{w} = [DL(\hat{w})]_{1\times 36}\,[p]_{36\times 1}
$$

where $[p]$ denotes the column matrix of basis polynomials. They can be written

$$
(3.1.43)\qquad [p]_{36\times 1} = [A]_{36\times 36}\,[m7]_{36\times 1}
$$

where $[m7]_{36\times 1}$ collects the 36 basis monomials of degrees less or equal to 7 with respect to $\hat{x}_1$and $\hat{x}_2$. From (3.1.42), we have

$$[m_7]_{36\times 1} = [DL(m7)]_{36\times 36}\,[p]_{36\times 1} \text{ so that } [A]_{36\times 36} = [DL(m7)]^{-1}.$$

These basis polynomials and the matrix $[A]$ are detailed in the annex of [10].

Finally, we get the interpolating function $\pi_K v$ of the function v by collecting relations (3.1.1) (3.1.11) (3.1.42) and (3.1.43) :

$$(3.1.44) \qquad \pi_K v(x) = \hat{w}(\hat{x}) = [DLGL(v)]_{1\times 24}\,[\tilde{D}]_{24\times 24}\,[B]_{24\times 36}\,[A]_{36\times 36}\,[m7]_{36\times 1}$$

3.2 Interpolation modulus associated to the curved finite element $\mathcal{C}^1$-compatible with ARGYRIS element when $F_K^* \in (P_5)^2$

The matrix decompositions are entirely similar to that of section 3.1 so that we just give the results and we send back to the above section for more details. Now

$$(3.2.1) \qquad \pi_K^* v(x) = w^*(x)$$

which is constructed in three steps from the application $F_K^* \in (P_5)^2$ defined by (1.3.6).

Step 1 : $v \longrightarrow \Sigma_K^*(v) = [DLLC^*(v)]$. From (2.3.2), we get :

$$(3.2.2) \quad \left\{ \begin{array}{l} [DLLC^*(v)]_{1\times 31} = [v(a_1)\;\; v(a_2)\;\; v(a_3)\;\; Dv(a_1)(a_3-a_1)\;\; (\bar{s}-\underline{s})Dv(a_1)\vec{\chi}'(\underline{s}) \\ \quad -(\bar{s}-\underline{s})Dv(a_2)\vec{\chi}'(\bar{s})\;\; Dv(a_2)(a_3-a_2)\;\; Dv(a_3)(a_2-a_3)\;\; Dv(a_3)(a_1-a_3) \\ \quad D^2v(a_1)(a_3-a_1)^2\;\; (\bar{s}-\underline{s})^2 D^2v(a_1)(\vec{\chi}'(\underline{s}))^2\;\; (\underline{s}-\bar{s})^2 D^2v(a_2)(\vec{\chi}'(\bar{s}))^2 \\ \quad D^2v(a_2)(a_3-a_2)^2\;\; D^2v(a_3)(a_2-a_3)^2\;\; D^2v(a_3)(a_1-a_3)^2\;\; D^2v(a_1)(a_2-a_3)^2 \\ \quad D^2v(a_2)(a_3-a_1)^2\;\; (\bar{s}-\underline{s})^2 D^2v(a_3)((\vec{\chi}'(\underline{s}),\vec{\chi}'(\bar{s}))\;\; Dv(b_1)(a_1-c_1) \\ \quad Dv(b_2)(a_2-c_2)\;\; Dv(b_3)DF_K^*(\hat{b}_3)(\hat{a}_3-\hat{b}_3)\;\; v(e_1^*)\;...\;v(e_{10}^*)] \end{array} \right.$$

while the set of global degrees of freedom is given by

$$(3.2.3) \quad \left\{ \begin{array}{l} [DLGL^*(v)]_{1\times 31} = [v(a_1)\;\; v(a_2)\;\; v(a_3)\;\; \dfrac{\partial v}{\partial x_1}(a_1)\;\; \dfrac{\partial v}{\partial x_2}(a_1)\;\; \dfrac{\partial v}{\partial x_1}(a_2) \\ \dfrac{\partial v}{\partial x_2}(a_2)\;\; \dfrac{\partial v}{\partial x_1}(a_3)\;\; \dfrac{\partial v}{\partial x_2}(a_3)\;\; \dfrac{\partial^2 v}{\partial x_1^2}(a_1)\;\; \dfrac{\partial^2 v}{\partial x_1 \partial x_2}(a_1)\;\; \dfrac{\partial^2 v}{\partial x_2^2}(a_1) \\ \dfrac{\partial^2 v}{\partial x_1^2}(a_2)\;\; \dfrac{\partial^2 v}{\partial x_1 \partial x_2}(a_2)\;\; \dfrac{\partial^2 v}{\partial x_2^2}(a_2)\;\; \dfrac{\partial^2 v}{\partial x_1^2}(a_3)\;\; \dfrac{\partial^2 v}{\partial x_1 \partial x_2}(a_3)\;\; \dfrac{\partial^2 v}{\partial x_2^2}(a_3) \\ \dfrac{\partial v}{\partial \nu_1}(b_1)\;\; \dfrac{\partial v}{\partial \nu_2}(b_2)\;\; Dv(b_3)DF_K^*(\hat{b}_3)(\hat{a}_3-\hat{b}_3)\;\; v(e_1^*)\;...\;v(e_{10}^*)] \end{array} \right.$$

Then, with (3.1.6), we get

$$(3.2.4) \qquad [DLLC^*(v)]_{1\times 31} = [DLGL^*(v)]_{1\times 31}\,[\tilde{D}^*]_{31\times 31}, \quad \tilde{D}^* = \begin{bmatrix} \tilde{D} & 0 \\ 0 & I_7 \end{bmatrix}$$

Step 2 : Transition from $\Sigma_K^*(v) = [DLLC^*(v)]$ to $\hat{\Delta}_K^*(v) = [DL^*(\hat{w}^*)]$: According to Figure 2.3.1, we set

$$(3.2.5) \qquad \left\{ \begin{array}{l} [DL^*(\hat{w}^*)]_{1\times 55} = [\hat{w}^*(\hat{a}_1) \;\; \hat{w}^*(\hat{a}_2) \;\; \hat{w}^*(\hat{a}_3) \;\; \dfrac{\partial \hat{w}^*}{\partial \hat{x}_1}(\hat{a}_1) \;\; \dfrac{\partial \hat{w}^*}{\partial \hat{x}_2}(\hat{a}_1) \\[2ex] \quad \dfrac{\partial \hat{w}^*}{\partial \hat{x}_1}(\hat{a}_2) \;\; \dfrac{\partial \hat{w}^*}{\partial \hat{x}_2}(\hat{a}_2) \;\; \dfrac{\partial \hat{w}^*}{\partial \hat{x}_1}(\hat{a}_3) \;\; \dfrac{\partial \hat{w}^*}{\partial \hat{x}_2}(\hat{a}_3) \;\; \dfrac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_1) \\[2ex] \quad \dfrac{\partial^2 \hat{w}^*}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_1) \;\; \dfrac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_1) \;\; \dfrac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_2) \;\; \dfrac{\partial^2 \hat{w}^*}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_2) \\[2ex] \quad \dfrac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_2) \;\; \dfrac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_3) \;\; \dfrac{\partial^2 \hat{w}^*}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_3) \;\; \dfrac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_3) \;\; -\dfrac{\partial \hat{w}^*}{\partial \hat{x}_1}(\hat{b}_1) \\[2ex] \quad -\dfrac{\partial \hat{w}^*}{\partial \hat{x}_2}(\hat{b}_2) \;\; \dfrac{\sqrt{2}}{2}\left(\dfrac{\partial \hat{w}^*}{\partial \hat{x}_1} + \dfrac{\partial \hat{w}^*}{\partial \hat{x}_2}\right)(\hat{b}_3) \;\; \{\hat{w}^*(\hat{d}_i^*),\ i = 1, \dots, 12\} \\[2ex] \quad \{-\dfrac{\partial \hat{w}^*}{\partial \hat{x}_1}(\hat{d}_i^*),\ i = 1, \dots, 4\} \;\; \{-\dfrac{\partial \hat{w}^*}{\partial \hat{x}_2}(\hat{d}_i^*),\ i = 5, \dots, 8\} \\[2ex] \quad \{\dfrac{\sqrt{2}}{2}\left(\dfrac{\partial \hat{w}^*}{\partial \hat{x}_1} + \dfrac{\partial \hat{w}^*}{\partial \hat{x}_2}\right)(\hat{d}_i^*),\ i = 9, \dots, 12\} \;\; \{\hat{w}(\hat{e}_i^*),\ i = 1, \dots, 10\}] \end{array} \right.$$

It remains to determine the transition matrix $[B^*]$, i.e.,

$$(3.2.6) \qquad [DL^*(\hat{w}^*)]_{1\times 55} = [DLLC^*(v)]_{1\times 31}\,[B^*]_{31\times 55}$$

so that

$$(3.2.7) \qquad [DL^*(\hat{w}^*)]_{1\times 55} = [DLGL^*(v)]_{1\times 31}\,[\tilde{D}^*]_{31\times 31}\,[B^*]_{31\times 55}.$$

To specify B^*, it is convenient to consider the following partition :

$$(3.2.8) \qquad [B^*]_{31\times 55} = [B_1^* \;\; B_2^* \;\; B_3^* \;\; B_4^* \;\; B_5^* \;\; B_6^* \;\; B_7^* \;\; B_8^* \;\; B_9^*]$$

where matrices B_i^*, $i = 1, \dots, 9$ have 31 lines and 3, 6, 9, 3, 12, 4, 4, 4, 10 columns.

Construction of submatrix B_1^* and B_9^* : Relations (2.3.8) and (2.3.9) involve $\hat{w}^*(\hat{a}_i) = w^*(a_i) = v(a_i)$, $i = 1, \dots, 3$ and $\hat{w}^*(\hat{e}_i^*) = w^*(e_i) = v(e_i)$, $i = 1, \dots, 10$, so that

$$(3.2.9) \qquad {}^tB_1^* = [I_{3\times 3}\ ;\ 0_{3\times 28}], \quad {}^tB_9^* = [0_{10\times 21}\ ;\ I_{10\times 10}].$$

Construction of submatrix B_2^* : Relations (3.1.14) are unchanged, except for the star (*). Then, from (1.3.6), relation (3.1.16) remains valid so that with (3.1.17), we get :

$$(3.2.10) \qquad {}^tB_2^* = [0_{6\times 3} \,;\, {}^t(b_2)_{6\times 6} \,;\, 0_{6\times 22}].$$

Construction of submatrix B_3^* : Relations (2.3.9) and (3.1.18) involve

$$(3.2.11) \qquad \frac{\partial^2 \hat{w}^*}{\partial \hat{x}_\alpha \partial \hat{x}_\beta}(\hat{a}_i) = D^2 v(a_i)\left(\frac{\partial F_K^*}{\partial \hat{x}_\alpha}(\hat{a}_i), \frac{\partial F_K^*}{\partial \hat{x}_\beta}(\hat{a}_i)\right) + Dv(a_i)\,\frac{\partial^2 F_K^*}{\partial \hat{x}_\alpha \partial \hat{x}_\beta}(\hat{a}_i)$$

so that, with (1.3.6),

$$(3.2.12) \quad \left\{ \begin{aligned}
\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_1) &= D^2 v(a_1)(a_1 - a_3)^2 \\
\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_1) &= D^2 v(a_1)[a_1 - a_3,\ a_1 - a_3 + (\bar{s}-\underline{s})\vec{\chi}'(\underline{s})] \\
&\quad + \frac{1}{2}\, Dv(a_1)\left[a_1 - a_2 + (\bar{s}-\underline{s})\vec{\chi}'(\underline{s}) - \frac{1}{2}(\bar{s}-\underline{s})^2 \vec{\chi}''(\underline{s})\right] \\
\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_1) &= D^2 v(a_1)[a_1 - a_3 + (\bar{s}-\underline{s})\vec{\chi}'(\underline{s})]^2 \\
&\quad + Dv(a_1)\left[a_1 - a_2 + (\bar{s}-\underline{s})\vec{\chi}'(\underline{s}) + \frac{(\bar{s}-\underline{s})^2}{2}\,\vec{\chi}''(\underline{s})\right] \\
\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_2) &= D^2 v(a_2)[a_2 - a_3 - (\bar{s}-\underline{s})\vec{\chi}'(\bar{s})]^2 \\
&\quad + Dv(a_2)\left[a_2 - a_1 - (\bar{s}-\underline{s})\vec{\chi}'(\bar{s}) + \frac{(\bar{s}-\underline{s})^2}{2}\,\vec{\chi}''(\bar{s})\right] \\
\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_2) &= D^2 v(a_2)[a_2 - a_3 - (\bar{s}-\underline{s})\vec{\chi}'(\bar{s}), a_2 - a_3] \\
&\quad + \frac{1}{2}\, Dv(a_2)\left[a_2 - a_1 - (\bar{s}-\underline{s})\vec{\chi}'(\bar{s}) - \frac{1}{2}(\bar{s}-\underline{s})^2 \vec{\chi}''(\bar{s})\right] \\
\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_2) &= D^2 v(a_2)[a_2 - a_3]^2 \\
\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_3) &= D^2 v(a_3)[a_1 - a_3]^2 \\
\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_3) &= D^2 v(a_3)[a_1 - a_3,\ a_2 - a_3] + \frac{1}{2}(\bar{s}-\underline{s})\, Dv(a_3)[\vec{\chi}'(\underline{s}) - \vec{\chi}'(\bar{s})] \\
\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_3) &= D^2 v(a_3)[a_2 - a_3]^2
\end{aligned} \right.$$

Then

$$(3.2.13) \qquad {}^tB_3^* = [0_{9\times 3} \,;\, {}^t(b_{31}^*)_{9\times 6} \,;\, {}^t(b_{32})_{9\times 9} \,;\, 0_{9\times 13}]$$

where matrix b_{32} is defined by (3.1.22) while b_{31}^* is given by (3.2.14).

$$
(3.2.14) \qquad b^*_{31} = \begin{bmatrix}
0 & \frac{1}{4}(2\tilde{a}^1 - \underset{\sim}{a}^1) & (\tilde{a}^1 + \frac{1}{2}\underset{\sim}{a}^1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4}(2 + 2\tilde{a}^2 - \underset{\sim}{a}^2) & 1 + \tilde{a}^2 + \frac{1}{2}\underset{\sim}{a}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (1 + \tilde{b}^1 + \frac{1}{2}\underset{\sim}{b}^1) & \frac{1}{4}(2 + 2\tilde{b}^1 - \underset{\sim}{b}^1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\tilde{b}^2 + \frac{1}{2}\underset{\sim}{b}^2) & \frac{1}{4}(2\tilde{b}^2 - \underset{\sim}{b}^2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\tilde{c}^1 + \tilde{\bar{c}}^1) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\tilde{c}^2 + \tilde{\bar{c}}^2) & 0
\end{bmatrix}
$$

In this expression, parameters $\tilde{a}^\alpha$, $\tilde{c}^\alpha$ and $\tilde{\bar{c}}^\alpha$ are given by relations (3.1.29) while parameters $\underset{\sim}{a}^\alpha$ and $\underset{\sim}{b}^\alpha$ are defined by

$$
(3.2.15) \qquad \begin{cases}
\underset{\sim}{a}^1 = \dfrac{(\vec{D}_1 \times \vec{A}_2).\vec{e}_3}{(\vec{A}_1 \times \vec{A}_2).\vec{e}_3} & ; \quad \underset{\sim}{a}^2 = \dfrac{(\vec{A}_1 \times \vec{D}_1).\vec{e}_3}{(\vec{A}_1 \times \vec{A}_2).\vec{e}_3} \\[2ex]
\underset{\sim}{b}^1 = \dfrac{(\vec{D}_2 \times \vec{B}_2).\vec{e}_3}{(\vec{B}_1 \times \vec{B}_2).\vec{e}_3} & ; \quad \underset{\sim}{b}^2 = \dfrac{(\vec{B}_1 \times \vec{D}_2).\vec{e}_3}{(\vec{B}_1 \times \vec{B}_2).\vec{e}_3} \\[2ex]
\vec{D}_1 = (\bar{s} - \underline{s})^2 \vec{\chi}''(\underline{s}) = \underset{\sim}{a}^\alpha \vec{A}_\alpha & ; \quad \vec{D}_2 = (\bar{s} - \underline{s})^2 \vec{\chi}''(\bar{s}) = \underset{\sim}{b}^\alpha \vec{B}_\alpha
\end{cases}
$$

Construction of submatrix B^*_4 : Since F_K and F^*_K are affine along the sides $[\hat{a}^*_3, \hat{a}^*_\alpha]$, $\alpha = 1, 2$, functions $\hat{f}_\alpha, \hat{g}_\alpha$, $\alpha = 1, 2$ are unchanged. By using (1.3.6), we obtain

$$
(3.2.16) \qquad \begin{cases}
\left[\hat{f}_1(\hat{x}_1) \;\; \hat{f}_2(\hat{x}_2) \;\; \hat{f}^*_3(\hat{x}_1)\right] & = [DLLC^*(v)]_{1\times 31} \, [\hat{\mathcal{F}}^*_1(\hat{x}_1) \;\; \hat{\mathcal{F}}^*_2(\hat{x}_2) \;\; \hat{\mathcal{F}}^*_3(\hat{x}_1)]_{31\times 3} \\[2ex]
[\hat{g}_1(\hat{x}_1) \;\; \hat{g}_2(\hat{x}_2) \;\; \hat{g}^*_3(\hat{x}_1)] & = [DLLC^*(v)]_{1\times 31} \, [\hat{\mathcal{G}}^*_1(\hat{x}_1) \;\; \hat{\mathcal{G}}^*_2(\hat{x}_2) \;\; \hat{\mathcal{G}}^*_3(\hat{x}_1)]_{31\times 3}
\end{cases}
$$

with

$$
(3.2.17) \qquad \left\{ \; {}^t[\hat{\mathcal{F}}^*_\alpha(\hat{x}_\alpha)] = [{}^t\hat{\mathcal{F}}_\alpha(\hat{x}_\alpha) \; ; \; 0_{1\times 7}] \; ; \; [{}^t\hat{\mathcal{G}}^*_\alpha(\hat{x}_\alpha)] = [{}^t\hat{\mathcal{G}}_\alpha(\hat{x}_\alpha) \; ; \; 0_{1\times 7}], \quad \alpha = 1, 2 \right.
$$

$$
(3.2.18) \qquad \begin{cases}
{}^t[\hat{\mathcal{F}}^*_3(\hat{x}_1)] = [(\hat{x}_1)^3\{6\hat{x}_1^2 - 15\hat{x}_1 + 10\} \; ; \; (1 - \hat{x}_1)^3\{6\hat{x}_1^2 + 3\hat{x}_1 + 1\} \; ; \\[2ex]
\quad 0 \; ; \; \frac{1}{2}\,\hat{x}_1^3(1 - \hat{x}_1)^2 \underset{\sim}{a}^1 \; ; \; \hat{x}_1^3(1 - \hat{x}_1)\{4 - 3\hat{x}_1 + \frac{1}{2}(1 - \hat{x}_1)\underset{\sim}{a}^2\} \; ; \\[2ex]
\quad \hat{x}_1(1 - \hat{x}_1)^3 \left\{1 + 3\hat{x}_1 + \frac{1}{2}\,\hat{x}_1 \underset{\sim}{b}^1\right\} \; ; \; \frac{1}{2}\,\hat{x}_1^2(1 - \hat{x}_1)^3 \underset{\sim}{b}^2 \; ; \; 0 \; 0 \; 0 \; ; \\[2ex]
\quad \frac{1}{2}\,\hat{x}_1^3(1 - \hat{x}_1)^2 \; ; \; \frac{1}{2}\,\hat{x}_1^2(1 - \hat{x}_1)^3 \; ; \; 0_{1\times 19}]
\end{cases}
$$

$$(3.2.19)\quad \left\{\begin{array}{l}
{}^t[\hat{\mathcal{G}}_3^*(\hat{x}_1)] = [0\;0\;0\;;\; \hat{x}_1^2(2\hat{x}_1-1)\left\{5-4\hat{x}_1-\frac{1}{2}(1-\hat{x}_1)\left(\tilde{a}^1+\frac{1}{2}\underset{\sim}{a}^1\right)\right\}\;; \\[2ex]
\hat{x}_1^2(2\hat{x}_1-1)\left\{-3+\frac{5\hat{x}_1}{2}-\frac{1}{2}(1-\hat{x}_1)\left(\tilde{a}^2+\frac{1}{2}\underset{\sim}{a}^2\right)\right\}\;; \\[2ex]
-\frac{1}{2}(1-\hat{x}_1)^2(1-2\hat{x}_1)\left\{1+5\hat{x}_1+\hat{x}_1(\tilde{b}^1+\frac{1}{2}\underset{\sim}{b}^1)\right\}\;; \\[2ex]
(1-\hat{x}_1)^2(1-2\hat{x}_1)\left\{1+4\hat{x}_1-\frac{1}{2}\hat{x}_1(\tilde{b}^2+\frac{1}{2}\underset{\sim}{b}^2)\right\}\;;\;0\;0\;; \\[2ex]
\frac{1+\tilde{a}^1}{2\tilde{a}^2}\,\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1)\;;\;\frac{1}{2}\left(1+\frac{\tilde{a}^2}{1+\tilde{a}^1}\right)\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1)\;; \\[2ex]
-\frac{1}{2}\left(1+\frac{\tilde{b}^1}{1+\tilde{b}^2}\right)\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1)\;;\;-\frac{1+\tilde{b}^2}{2\tilde{b}^1}\,\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1)\;;\;0\;0\;; \\[2ex]
-\frac{1}{2(1+\tilde{a}^1)\tilde{a}^2}\,\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1)\;;\;\frac{1}{2\tilde{b}^1(1+\tilde{b}^2)}\,\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1)\;; \\[2ex]
0\;0\;0\;;\;16(1-\hat{x}_1)^2\hat{x}_1^2\;;\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0]
\end{array}\right.$$

Then, by similarity with relation (3.1.37), we obtain

$$(3.2.20)\quad B_4^* = \left[\tilde{E}^1\,\frac{d}{d\hat{x}_2}\,\hat{\mathcal{F}}_2^*\left(\frac{1}{2}\right)+\tilde{E}^2\,\hat{\mathcal{G}}_2^*\left(\frac{1}{2}\right)\;;\;\tilde{F}^1\,\frac{d}{d\hat{x}_1}\,\hat{\mathcal{F}}_1^*\left(\frac{1}{2}\right)+\tilde{F}^2\hat{\mathcal{G}}_1^*\left(\frac{1}{2}\right)\;;\;\begin{pmatrix}0_{20\times1}\\ -\sqrt{2}\\ 0_{10\times3}\end{pmatrix}\right]_{31\times3}$$

with

$$\tilde{E}^1 =<a_3-a_1-\frac{1}{4}(a_2-a_1)-\frac{1}{32}(\bar{s}-\underset{=}{s})[5\vec{\chi}'(\underset{=}{s})-13\vec{\chi}'(\bar{s})]-\frac{(\bar{s}-\underset{=}{s})^2}{64}[\vec{\chi}''(\underset{=}{s})+\vec{\chi}''(\bar{s})],\;\frac{a_2-a_3}{|a_2-a_3|^2}>$$

$$\tilde{E}^2 =<a_3-a_1-\frac{1}{4}(a_2-a_1)-\frac{1}{32}(\bar{s}-\underset{=}{s})[5\vec{\chi}'(\underset{=}{s})-13\vec{\chi}'(\bar{s})]-\frac{(\bar{s}-\underset{=}{s})^2}{64}[\vec{\chi}''(\underset{=}{s})+\vec{\chi}''(\bar{s})],\;\frac{a_1-c_1}{|a_1-c_1|^2}>$$

$$\tilde{F}^1 =<a_3-a_2-\frac{1}{4}(a_1-a_2)-\frac{1}{32}(\bar{s}-\underset{=}{s})(5\vec{\chi}'(\bar{s})-13\vec{\chi}'(\underset{=}{s}))+\frac{(\bar{s}-\underset{=}{s})^2}{64}[\vec{\chi}''(\underset{=}{s})+\vec{\chi}''(\bar{s})],\;\frac{a_1-a_3}{|a_1-a_3|^2}>$$

$$\tilde{F}^2 =<a_3-a_2-\frac{1}{4}(a_1-a_2)-\frac{1}{32}(\bar{s}-\underset{=}{s})(5\vec{\chi}'(\bar{s})-13\vec{\chi}'(\underset{=}{s}))+\frac{(\bar{s}-\underset{=}{s})^2}{64}[\vec{\chi}''(\underset{=}{s})+\vec{\chi}''(\bar{s})],\;\frac{a_2-c_2}{|a_2-c_2|^2}>.$$

Construction of submatrix B_5^* : Relations (2.3.3) (2.3.6) (2.3.7) and (3.2.21) give

$$(3.2.21)\quad\left\{\begin{array}{l}
B_5^* = \left[\hat{\mathcal{F}}_2^*\left(\frac{5}{6}\right)\;;\;\hat{\mathcal{F}}_2^*\left(\frac{2}{3}\right)\;;\;\hat{\mathcal{F}}_2^*\left(\frac{1}{3}\right)\;;\;\hat{\mathcal{F}}_2^*\left(\frac{1}{6}\right)\;;\;\hat{\mathcal{F}}_1^*\left(\frac{1}{6}\right)\;;\;\hat{\mathcal{F}}_1^*\left(\frac{1}{3}\right)\;;\right. \\[2ex]
\left.\hat{\mathcal{F}}_1^*\left(\frac{2}{3}\right)\;;\;\hat{\mathcal{F}}_1^*\left(\frac{5}{6}\right)\;;\;\hat{\mathcal{F}}_3^*\left(\frac{5}{6}\right)\;;\;\hat{\mathcal{F}}_3^*\left(\frac{2}{3}\right)\;;\;\hat{\mathcal{F}}_3^*\left(\frac{1}{3}\right)\;;\;\hat{\mathcal{F}}_3^*\left(\frac{1}{6}\right)\right]_{31\times12}
\end{array}\right.$$

Construction of submatrices B_6^* and B_7^* : From relations (2.3.3) (2.3.6) (2.3.7) and (3.2.16), we obtain

$$(3.2.22)\quad \begin{cases} B_6^* = \Big[- < \dfrac{\partial F_K^*}{\partial \hat{x}_1}(\hat{d}_1^*), \dfrac{a_2-a_3}{|a_2-a_3|^2}\dfrac{d}{d\hat{x}_2}\hat{\mathcal{F}}_2^*\left(\dfrac{5}{6}\right) + \dfrac{a_1-c_1}{|a_1-c_1|^2}\hat{\mathcal{G}}_2^*\left(\dfrac{5}{6}\right) > ; \\ \quad - < \dfrac{\partial F_K^*}{\partial \hat{x}_1}(\hat{d}_2^*), \dfrac{a_2-a_3}{|a_2-a_3|^2}\dfrac{d}{d\hat{x}_2}\hat{\mathcal{F}}_2^*\left(\dfrac{2}{3}\right) + \dfrac{a_1-c_1}{|a_1-c_1|^2}\hat{\mathcal{G}}_2^*\left(\dfrac{2}{3}\right) > ; \\ \quad - < \dfrac{\partial F_K^*}{\partial \hat{x}_1}(\hat{d}_3^*), \dfrac{a_2-a_3}{|a_2-a_3|^2}\dfrac{d}{d\hat{x}_2}\hat{\mathcal{F}}_2^*\left(\dfrac{1}{3}\right) + \dfrac{a_1-c_1}{|a_1-c_1|^2}\hat{\mathcal{G}}_2^*\left(\dfrac{1}{3}\right) > ; \\ \quad - < \dfrac{\partial F_K^*}{\partial \hat{x}_1}(\hat{d}_4^*), \dfrac{a_2-a_3}{|a_2-a_3|^2}\dfrac{d}{d\hat{x}_2}\hat{\mathcal{F}}_2^*\left(\dfrac{1}{6}\right) + \dfrac{a_1-c_1}{|a_1-c_1|^2}\hat{\mathcal{G}}_2^*\left(\dfrac{1}{6}\right) > \Big]_{31\times 4} \end{cases}$$

$$(3.2.23)\quad \begin{cases} B_7^* = \Big[- < \dfrac{\partial F_K^*}{\partial \hat{x}_2}(\hat{d}_5^*), \dfrac{a_1-a_3}{|a_1-a_3|^2}\dfrac{d\hat{\mathcal{F}}_1^*}{d\hat{x}_1}\left(\dfrac{1}{6}\right) + \dfrac{a_2-c_2}{|a_2-c_2|^2}\hat{\mathcal{G}}_1^*\left(\dfrac{1}{6}\right) > ; \\ \quad - < \dfrac{\partial F_K^*}{\partial \hat{x}_2}(\hat{d}_6^*), \dfrac{a_1-a_3}{|a_1-a_3|^2}\dfrac{d\hat{\mathcal{F}}_1^*}{d\hat{x}_1}\left(\dfrac{1}{3}\right) + \dfrac{a_2-c_2}{|a_2-c_2|^2}\hat{\mathcal{G}}_1^*\left(\dfrac{1}{3}\right) > ; \\ \quad - < \dfrac{\partial F_K^*}{\partial \hat{x}_2}(\hat{d}_7^*), \dfrac{a_1-a_3}{|a_1-a_3|^2}\dfrac{d\hat{\mathcal{F}}_1^*}{d\hat{x}_1}\left(\dfrac{2}{3}\right) + \dfrac{a_2-c_2}{|a_2-c_2|^2}\hat{\mathcal{G}}_1^*\left(\dfrac{2}{3}\right) > ; \\ \quad - < \dfrac{\partial F_K^*}{\partial \hat{x}_2}(\hat{d}_8^*), \dfrac{a_1-a_3}{|a_1-a_3|^2}\dfrac{d\hat{\mathcal{F}}_1^*}{d\hat{x}_1}\left(\dfrac{5}{6}\right) + \dfrac{a_2-c_2}{|a_2-c_2|^2}\hat{\mathcal{G}}_1^*\left(\dfrac{5}{6}\right) > \Big] \end{cases}$$

$$(3.2.24)\quad B_8^* = \left[- \sqrt{2}\hat{\mathcal{G}}_3^*\left(\frac{5}{6}\right) ; - \sqrt{2}\hat{\mathcal{G}}_3^*\left(\frac{2}{3}\right) ; - \sqrt{2}\hat{\mathcal{G}}_3^*\left(\frac{1}{3}\right) ; - \sqrt{2}\hat{\mathcal{G}}_3^*\left(\frac{1}{6}\right)\right]$$

Step 3 : Transition from $\hat{\Delta}_K^*(v)$ to $\pi_K^* v(x) = w^*(x) = \hat{w}^*(\hat{x})$: Let $[m9]_{55\times 1}$ be the 55 basis monomials of degree less or equal to 9 in $\hat{x}_1$ and $\hat{x}_2$ and $[A^*]_{55\times 55} = [DL^*(m9)]^{-1}$. Corresponding basis polynomials are detailed in the annex of [10]. Then,

$$(3.2.25)\quad \pi_K^* v(x) = \hat{w}^*(\hat{x}) = [DLGL^*(v)]_{1\times 31}\, [\tilde{D}^*]_{31\times 31}\, [B^*]_{31\times 55}\, [A^*]_{55\times 55}\, [m9]_{55\times 1}.$$

3.3 Some numerical tests of interpolation properties

It is important to test numerically the interpolation properties. We will compute the asymptotic order of the interpolation error for the following two cases :
- case of an exact representation of the curved boundary ;
- case of an approximate representation of the curved boundary.

3.3.1 Tests on the asymptotic order of interpolation error (generalities)

From [2, (4.1) and (4.51)], we have to numerically verify that

$$(3.3.1)\quad \begin{cases} |v - \pi_K v|_{m,K} \le c h_K^{6-m} \|v\|_{6,K} \\ |v - \pi_K^* v|_{m,K} \le c h_K^{6-m} \|v\|_{6,K} \end{cases}, \quad m = 0, \dots 6, \ \forall v \in H^6(K).$$

For simplicity, since it can be proved that the asymptotic error estimates in norms $|.|_{L^2(K)}$ and $|.|_{L^\infty(K)}$ have the same order, we will just check that

$$(3.3.2) \qquad |v - \pi_K v|_{0,\infty,K} = 0(h_K^6), \quad |v - \pi_K^* v|_{0,\infty,K} = 0(h_K^6), \quad \forall v \in W^{6,\infty}(K),$$

or, more easily, that the asymptotic behaviour of $|v - \pi_K v|$ (resp. $|v - \pi_K^* v|$) at a given point $F_{K_h}(\hat{x}_0, \hat{x}_0)$ (resp. $F_{K_h}^*(\hat{x}_0, \hat{x}_0)$) is in $O(h_h^6)$.

In this way, for each example into consideration (see §§ 3.3.2 and 3.3.3), we successively

i) consider a series of nested triangles whose sizes decrease as $h, \frac{h}{2}, \frac{h}{4}, ...,$;

ii) interpolate given regular functions v ;

iii) illustrate the results on graphics displaying $-$ *Log* h in abcissa and $-$ *Log* $|v - \pi_K v|$ (resp. $-$ *Log* $|v - \pi_K^* v|$) in ordinate and check that

$$(3.3.3) \qquad Log\ |v - \pi_K v| = C + 6 Log\ h \ ; \ Log\ |v - \pi_K^* v| = C + 6 Log\ h.$$

3.3.2 Numerical tests for an "exact" interpolation of the boundary

Firstly it seems interesting to make some numerical experiments for an exact representation of the curved boundary so that it only remains the interpolation error of the function.

Example 3.3.1 : Unit right-angled triangle

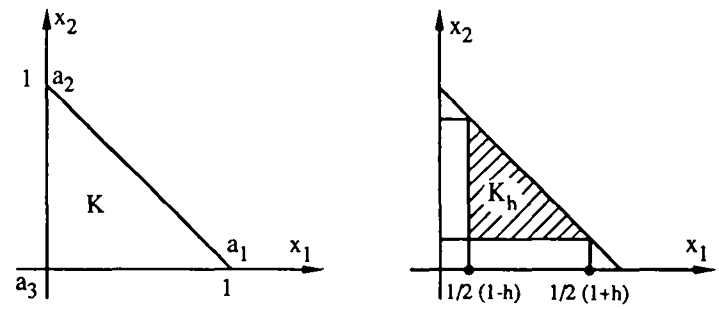


Figure 3.3.1 : The unit right-angled triangle and the associate sequence of triangles K_h

Of course this is the most simple case which is associated to the following representation of the curved (here straight !) side a_1a_2 :

$$(3.3.4) \qquad x_1 = \chi_1(s) = 1 - s, \quad x_2 = \chi_2(s) = s \quad \text{with } \underline{s} = 0 \leq s \leq \bar{s} = 1.$$

This test is interesting since $F_K = I$ and $F_K^* = I$ involves

$$(3.3.5) \qquad \pi_K v(x) \equiv v(x), \quad \pi_K^* v(x) = v(x), \quad \forall v \in P_5$$

or, equivalently, by using (3.1.44) and (3.2.25)

$$(3.3.6) \qquad \begin{cases} x_1^p x_2^q = [DLGL(x_1^p x_2^q)]\,[\tilde{D}]\,[B]\,[A]\,[m7], \quad \forall p,q \in \mathbb{N} \text{ and } p+q=5\,; \\ x_1^p x_2^q = [DLGL^*(x_1^p x_2^q)]\,[\tilde{D}^*]\,[B^*]\,[A^*]\,[m9], \quad \forall p,q \in \mathbb{N} \text{ and } p+q=5. \end{cases}$$

The numerical results show that identities (3.3.5) are exactly satisfied. We have also checked the estimates (3.3.3) ; in this way, we have considered a sequence of right-angled triangle deduced from the unit right-angled triangle by an homothetic transformation centered at point $(\frac{1}{2}, \frac{1}{2})$ and whose ratio is $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ (see Figure 3.3.1). More generally, we have successfully checked the same properties for any straight triangle K (note that the associate mapping F_K is still affine in this case).

Example 3.3.2 : A curved unit right-angled triangle (case $F_K \in (P_3)^2$)

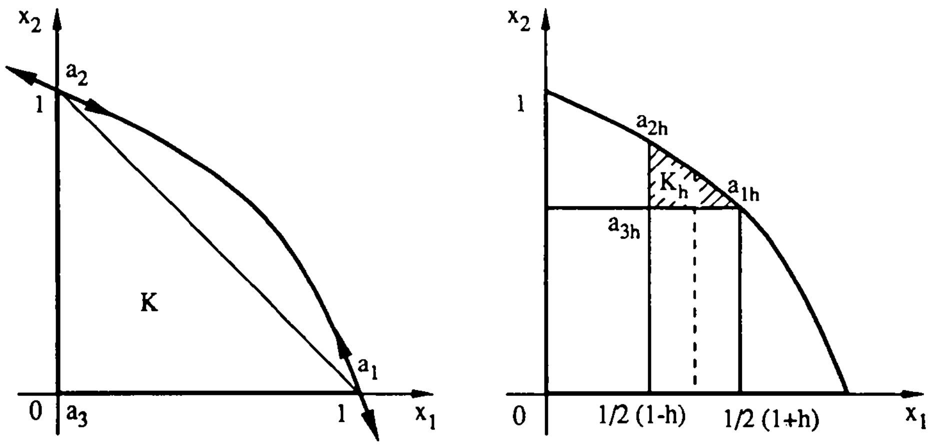


Figure 3.3.2 : The curved unit right-angled triangle and the associate sequence of triangles K_h

Now the straight side $a_1 a_2$ is replaced by the curved side whose parametric equations are (see Figure 3.3.2) :

$$(3.3.7) \qquad x_1 = \chi_1(s) = 1 - s, \quad x_2 = \chi_2(s) = \frac{1}{2}\, s[s^2 - 3s + 4], \quad \underline{s} = 0 \le s \le \bar{s} = 1,$$

so that its interpolation by the mapping (1.3.2) is exact. The associate application F_K is given by (see (1.3.2)) :

$$(3.3.8) \qquad F_{K1}(\hat{x}_1, \hat{x}_2) = \hat{x}_1, \quad F_{K2}(\hat{x}_1, \hat{x}_2) = \hat{x}_2 + \frac{1}{4}\, \hat{x}_1 \hat{x}_2 [3 + \hat{x}_1 - \hat{x}_2].$$

Now, since the mapping F_K is no longer affine, the identities (3.3.5) are no longer satisfied. In order to test the estimate $(3.3.2)_1$, we have to introduce a sequence of triangle K_h whose diameter h is decreasing to 0 and whose curved sides coincide (for this particular example) with the curved boundary of the domain. In this way we will use the triangles K_h

whose vertices (a_{1h}, a_{2h}, a_{3h}) are given by :

$$(3.3.9)\quad \begin{cases} \text{Vertex } a_{1h} : \underline{s}_h = \dfrac{1}{2}\,(1-h) \Longrightarrow \left(x_{11} = \dfrac{1}{2}\,(1+h)\,;\; x_{21} = \dfrac{1}{16}\,(1-h)(11+4h+h^2)\right) \\ \text{Vertex } a_{2h} : \bar{s}_h = \dfrac{1}{2}\,(1+h) \Longrightarrow \left(x_{12} = \dfrac{1}{2}\,(1-h)\,;\; x_{22} = \dfrac{1}{16}\,(1+h)(11-4h+h^2)\right) \\ \text{Vertex } a_{3h} : \left(x_{13} = \dfrac{1}{2}\,(1-h)\,;\; x_{23} = \dfrac{1}{16}\,(1-h)(11+4h+h^2)\right) \end{cases}$$

so that the associate application F_{K_h} is given by relation (1.3.2), i.e.,

$$(3.3.10)\quad \begin{cases} F_{K1_h}(\hat{x}_1, \hat{x}_2) &= \dfrac{1}{2}\,(1-h) + h\hat{x}_1 \\ F_{K2_h}(\hat{x}_1, \hat{x}_2) &= \dfrac{1}{16}\,(1-h)(11+4h+h^2) + \dfrac{h}{8}\,(7+h^2)\hat{x}_2 \\ &\quad + \dfrac{1}{4}\,\hat{x}_1\hat{x}_2 h^2[3 + h(\hat{x}_1 - \hat{x}_2)]. \end{cases}$$

Then, to check estimate $(3.3.2)_1$, we graphically represent $(3.3.3)_1$ for given functions v, for instance $v(x_1, x_2) = x_1^6 x_2^4$, and $v(x_1, x_2) = cos(3x_1)sin(2x_2)$, at the image of point $(\hat{x}_1 = \frac{3}{8},\ \hat{x}_2 = \frac{3}{8})$ by the application F_{K_h}, i.e.

$$(3.3.11)\quad x_1 = F_{K1_h}\left(\frac{3}{8}, \frac{3}{8}\right) = \frac{1}{2} - \frac{h}{8}\ ;\ x_2 = F_{K2_h}\left(\frac{3}{8}, \frac{3}{8}\right) = \frac{11}{16} - \frac{7h}{64} - \frac{21h^2}{256} - \frac{h^3}{64}$$

for $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ Thus, Figure 3.3.3 shows that the slope is effectively 6 according to the theoretical prediction (3.3.3).

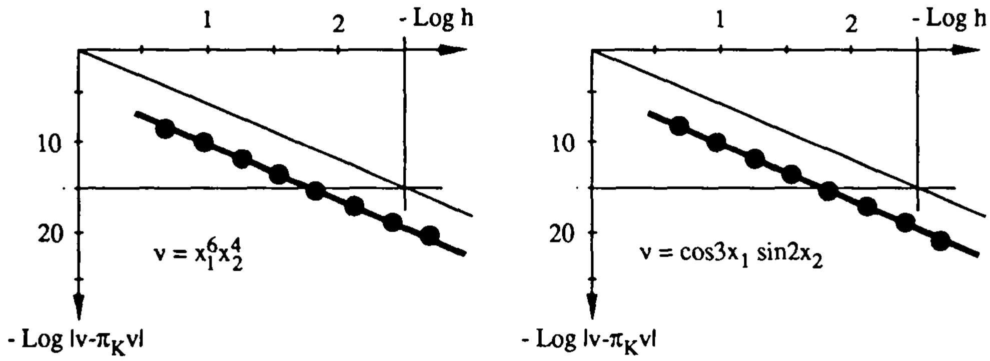


Figure 3.3.3 : Asymptotic behaviour of $|v - \pi_K v|$ at point $F_{K_h}(\frac{3}{8}, \frac{3}{8})$ when $h \longrightarrow 0$.

Example 3.3.3 : A curved unit right-angled triangle (case $F_K^* \in (P_5)^2$) and the associate sequence of triangles K_h

Consider the curved side a_1a_2 whose parametric equations are (see Figure 3.3.4)

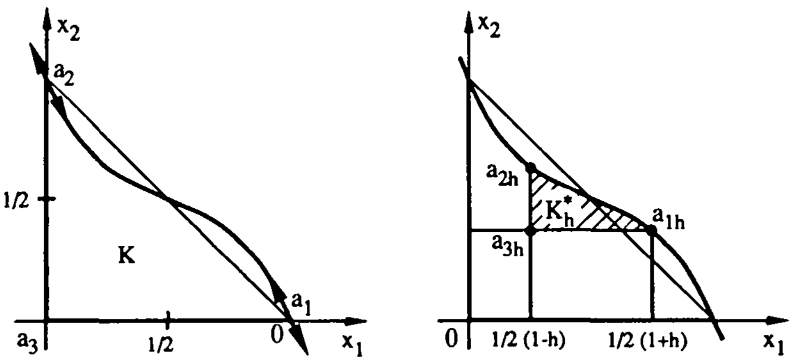


Figure 3.3.4 : The curved unit right-angled triangle and the associate sequence of triangles K_h^*

$$(3.3.12)\qquad \begin{cases} x_1 = \chi_1(s) = 1 - s, \quad x_2 = \chi_2(s) = \dfrac{s}{10}\,[4s^4 + 13s^2 - 27s + 20], \\[2mm] \underline{s} = 0 \le s \le \bar{s} = 1 \end{cases}$$

so that its interpolation by the mapping (1.3.6) is exact. The associate application F_K^* is given by (see (1.3.6)) :

$$(3.3.13)\qquad \begin{cases} F_{K1}^*(\hat{x}_1, \hat{x}_2) = \hat{x}_1 \\[2mm] F_{K2}^*(\hat{x}_1, \hat{x}_2) = \hat{x}_2 + \dfrac{1}{20}\,\hat{x}_1\hat{x}_2[-4(\hat{x}_2)^3 - 4(\hat{x}_2)^2 - 17\hat{x}_2 - 5 \\[2mm] \qquad + 4(\hat{x}_1)^3 - 16(\hat{x}_1)^2 + 37\hat{x}_1] \end{cases}$$

Here again we consider the sequence of triangles K_h whose vertices are given by

$$(3.3.14)\qquad \begin{cases} \text{vertex } a_{1h} : \left(x_{11} = \dfrac{1}{2}\,(1+h)\;;\; x_{21} = \dfrac{1}{80}\,(1-h)(40+24h+19h^2-4h^3+h^4)\right) \\[3mm] \text{vertex } a_{2h} : \left(x_{12} = \dfrac{1}{2}\,(1-h)\;;\; x_{22} = \dfrac{1}{80}\,(1+h)(40-24h+19h^2+4h^3+h^4)\right) \\[3mm] \text{vertex } a_{3h} : \left(x_{13} = \dfrac{1}{2}\,(1-h)\;;\; x_{23} = \dfrac{1}{80}\,(1-h)(40+24h+19h^2-4h^3+h^4)\right) \end{cases}$$

so that the application $F_{K_h}^*$ is given by relation (1.3.6) :

$$
(3.3.15)\quad \begin{cases}
F^*_{K1_h}(\hat{x}_1,\hat{x}_2) = \frac{1}{2}\,(1-h) + h\hat{x}_1 \\[2ex]
F^*_{K2_h}(\hat{x}_1,\hat{x}_2) = \frac{1}{80}\,(1-h)(40+24h+19h^2-4h^3+h^4) \\[2ex]
+\frac{h}{40}\,(16+23h^2+h^4)\hat{x}_2 + \frac{1}{2}\,\hat{x}_1\hat{x}_2[\frac{h^2}{2}\,(1-2h^2) \\[2ex]
+\frac{h^3}{10}\,(23+10h+4h^2)\hat{x}_1 + \frac{h^3}{10}\,(-23+10h-4h^2)\hat{x}_2 \\[2ex]
-\frac{h^4}{5}\,(5+3h)(\hat{x}_1)^2 - \frac{h^4}{5}\,(5-3h)(\hat{x}_2)^2 + \frac{2h^5}{5}\,(\hat{x}_1)^3 - \frac{2h^5}{5}\,(\hat{x}_2)^3]
\end{cases}
$$

Now we obtain Figure 3.3.5 which shows again an $0(h^6)$ error estimate.

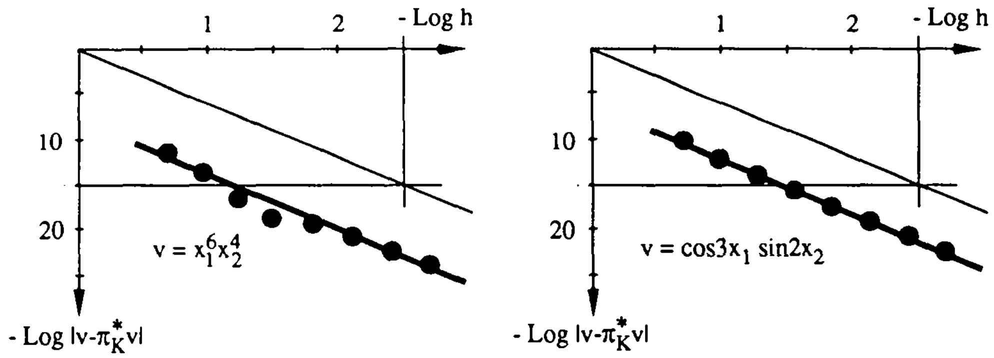


Figure 3.3.5 : Asymptotic behaviour of $|v-\pi^*_K v|$ at point $F^*_{K_h}(\frac{1}{4},\frac{1}{4})$ when $h \longrightarrow 0$.

3.3.3 Numerical tests for an approximate interpolation of the boundary

Previous examples are such that applications $F_K \in (P_3)^2$ or $F^*_K \in (P_5)^2$ give an exact representation of the curved side a_1a_2. Now let us consider some examples for which the curved side is only approximated by the applications F_K or F^*_K.

Example 3.3.4 : The $(P_3)^2$ approximation of the curved unit right-angled triangle studied in the example 3.3.3

In the example 3.3.3 we have considered a sequence of triangles K^*_h such that the curved side $a_{1h}a_{2h}$ coincided with the initial boundary. In this way we have used the application $F^*_{K_h} \in (P_5)^2$ defined by relation (3.3.15). Now, starting from the same sequence of curved triangle $a_{1h}a_{2h}a_{3h}$ we consider the $(P_3)^2$-approximation of the curved side. Thus, by substituting relations (3.3.12) (3.3.14) into relations (1.3.2), we get the application F_{K_h}

$$
(3.3.16)\begin{cases}
F_{K1_h}(\hat{x}_1,\hat{x}_2) = \dfrac{1}{2}\,(1-h) + h\hat{x}_1 \\[2ex]
F_{K2_h}(\hat{x}_1,\hat{x}_2) = \dfrac{1}{80}\,(1-h)(40+24h+19h^2-4h^3+\ h^4) \\[2ex]
\quad + \dfrac{h}{40}\,(16+23h^2+h^4)\hat{x}_2\ +\ \dfrac{h^2}{20}\,\hat{x}_1\hat{x}_2[5-10h^2+h(23+2h^2)(\hat{x}_1-\hat{x}_2)]
\end{cases}
$$

Here again Figure 3.3.6 shows an $0(h^6)$ error estimate

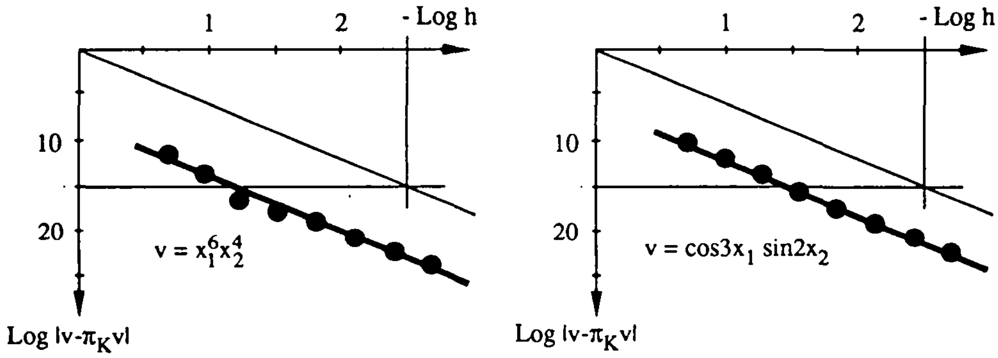


Figure 3.3.6 : Asymptotic behaviour of $|v - \pi_K v|$ at point $F_{K_h}(\frac{3}{8},\frac{3}{8})$ when $h \longrightarrow 0$.

Example 3.3.5 : Case of an elliptic edge

The curved side of triangle K is represented by (see Figure 3.3.7)

$$
(3.3.17) \qquad x_1 = \chi_1(s) = cos\ s\ ;\ x_2 = \chi_2(s) = R\ sin\ s,\ \ 0 < R < 1,\ \ \underline{s} = \frac{\pi}{6} \le s \le \bar{s} = \frac{\pi}{3},
$$

and it can be approximated with the help of $F_K \in (P_3)^2$ or $F_K^* \in (P_5)^2$, respectively defined by relations (1.3.2) or (1.3.6).

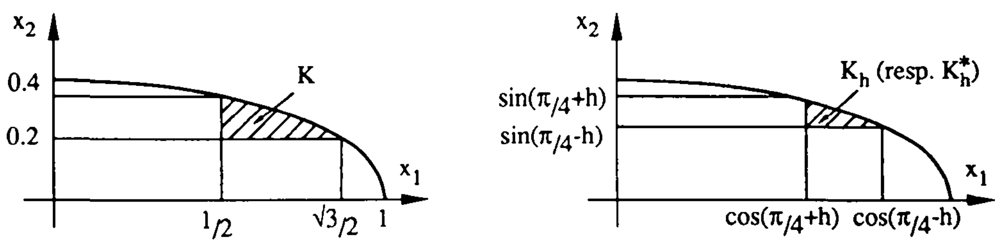


Figure 3.3.7 : The "curved elliptic" triangle K and the associate sequence of triangles K_h (or K_h^*) when $R = 0.4$

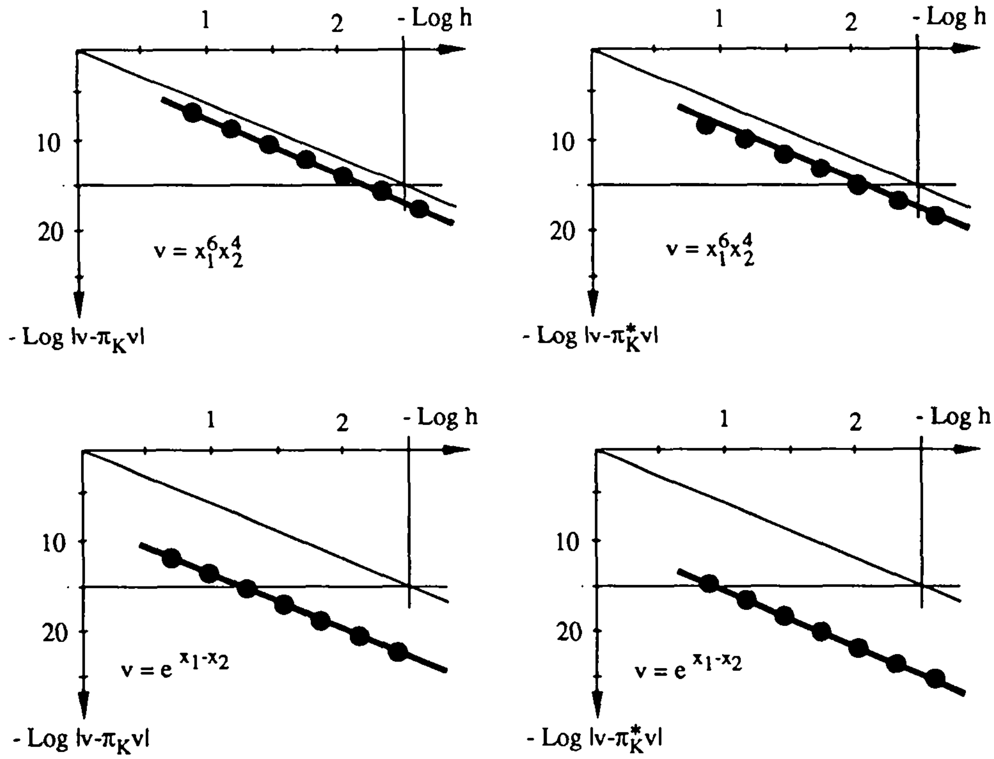


Figure 3.3.8 : Asymptotic behaviour of $|v - \pi_K v|$(resp. $|v - \pi_K^* v|$) at point $F_{K_h}(\frac{3}{8}, \frac{3}{8})$ (resp. $F_{K_h}^*(\frac{1}{4}, \frac{1}{4})$) when $h \longrightarrow 0$.

Here again we introduce the sequence K_h (or K_h^*) of triangles whose vertices are given by $(0 < h < \frac{\pi}{12})$:

$$
(3.3.18) \quad \begin{cases} \text{vertex } a_{1h} : \underline{s}_h = \dfrac{\pi}{4} - h \Longrightarrow \left(x_1 = cos\left(\dfrac{\pi}{4} - h\right) \; ; \; x_2 = R \; sin\left(\dfrac{\pi}{4} - h\right)\right) \\ \text{vertex } a_{2h} : \bar{s}_h = \dfrac{\pi}{4} + h \Longrightarrow \left(x_1 = cos\left(\dfrac{\pi}{4} + h\right) \; ; \; x_2 = R \; sin\left(\dfrac{\pi}{4} + h\right)\right) \\ \text{vertex } a_{3h} : \left(x_1 = cos\left(\dfrac{\pi}{4} + h\right), \; x_2 = R \; sin\left(\dfrac{\pi}{4} - h\right)\right) \end{cases}
$$

while the computation of the error is made at an internal point of triangles K_h (or K_h^*). The associated results are displayed on Figure 3.3.8. Here again, we find an asymptotic error in $0(h^6)$.

Concluding remarks

Thus, these examples show the great efficiency of these curved C^1 elements and they illustrate the relevance of the theoretical study of the order of the approximation. These methods will be used to approximate thin plate and thin shell problems in the second part of this work [10].

Acknowledgments

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ANNEX : Basis polynomials for P_7 and P_9 finite elements

In order to construct curved finite elements C^1-compatible with the Argyris triangle, we have proposed two methods :

(i) The first one is based in the interpolation of the boundary by polynomials of degree 3. In this way, we are led to the basic finite element displayed on Figure 2.2.1. Corresponding basis polynomials are given by relation (3.1.43), e.g.,

$$[p]_{36\times 1} = [A]_{36\times 36}\,[m7]_{36\times 1}.$$

The matrices $[m7]$ and $[A]$ are described on Figure A.1 and A.2.

(ii) The second one is based on the interpolation of the boundary by polynomials of degree 5. Then, we are led to the basic finite element which is shown on Figure 2.3.1. Corresponding basis polynomials are given by relation (3.2.25), e.g.,

$$[p^*]_{55\times 1} = [A^*]_{55\times 55}\,[m9]_{55\times 1}.$$

The associate matrices $[m9]$ and $[A^*]$ are given on Figure A.1 and A.3.

$$[m7] = \begin{bmatrix} \texttt{X1**7} \\ \texttt{X1**6 * X2} \\ \texttt{X1**5 * X2**2} \\ \texttt{X1**4 * X2**3} \\ \texttt{X1**3 * X2**4} \\ \texttt{X1**2 * X2**5} \\ \texttt{X1 * X2**6} \\ \texttt{X2**7} \\ \texttt{X1**6} \\ \texttt{X1**5 * X2} \\ \texttt{X1**4 * X2**2} \\ \texttt{X1**3 * X2**3} \\ \texttt{X1**2 * X2**4} \\ \texttt{X1 * X2**5} \\ \texttt{X2**6} \\ \texttt{X1**5} \\ \texttt{X1**4 * X2} \\ \texttt{X1**3 * X2**2} \\ \texttt{X1**2 * X2**3} \\ \texttt{X1 * X2**4} \\ \texttt{X2**5} \\ \texttt{X1**4} \\ \texttt{X1**3 * X2} \\ \texttt{X1**2 * X2**2} \\ \texttt{X1 * X2**3} \\ \texttt{X2**4} \\ \texttt{X1**3} \\ \texttt{X1**2 * X2} \\ \texttt{X1 * X2**2} \\ \texttt{X2**3} \\ \texttt{X1**2} \\ \texttt{X1 * X2} \\ \texttt{X2**2} \\ \texttt{X1} \\ \texttt{X2} \\ \texttt{1.} \end{bmatrix}$$

$$[m9] = \begin{bmatrix} \texttt{X1**9} \\ \texttt{X1**8 * X2} \\ \texttt{X1**7 * X2**2} \\ \texttt{X1**6 * X2**3} \\ \texttt{X1**5 * X2**4} \\ \texttt{X1**4 * X2**5} \\ \texttt{X1**3 * X2**6} \\ \texttt{X1**2 * X2**7} \\ \texttt{X1 * X2**8} \\ \texttt{X2**9} \\ \texttt{X1**8} \\ \texttt{X1**7 * X2} \\ \texttt{X1**6 * X2**2} \\ \texttt{X1**5 * X2**3} \\ \texttt{X1**4 * X2**4} \\ \texttt{X1**3 * X2**5} \\ \texttt{X1**2 * X2**6} \\ \texttt{X1 * X2**7} \\ \texttt{X2**8} \\ \texttt{X1**7} \\ \texttt{X1**6 * X2} \\ \texttt{X1**5 * X2**2} \\ \texttt{X1**4 * X2**3} \\ \texttt{X1**3 * X2**4} \\ \texttt{X1**2 * X2**5} \\ \texttt{X1 * X2**6} \\ \texttt{X2**7} \\ \texttt{X1**6} \\ \texttt{X1**5 * X2} \\ \texttt{X1**4 * X2**2} \\ \texttt{X1**3 * X2**3} \\ \texttt{X1**2 * X2**4} \\ \texttt{X1 * X2**5} \\ \texttt{X2**6} \\ \texttt{X1**5} \\ \texttt{X1**4 * X2} \\ \texttt{X1**3 * X2**2} \\ \texttt{X1**2 * X2**3} \\ \texttt{X1 * X2**4} \\ \texttt{X2**5} \\ \texttt{X1**4} \\ \texttt{X1**3 * X2} \\ \texttt{X1**2 * X2**2} \\ \texttt{X1 * X2**3} \\ \texttt{X2**4} \\ \texttt{X1**3} \\ \texttt{X1**2 * X2} \\ \texttt{X1 * X2**2} \\ \texttt{X2**3} \\ \texttt{X1**2} \\ \texttt{X1 * X2} \\ \texttt{X2**2} \\ \texttt{X1} \\ \texttt{X2} \\ \texttt{1.} \end{bmatrix}$$

Figure A.1 : Basis monomials of P_7 and P_9

```
A(1,1) =6496/27
A(1,3) =55376/27
A(1,4) =15296/3
A(1,5) =86464/27
A(1,6) =464/3
A(1,9) =(-6896)/9
A(1,11) =(-136424)/27
A(1,12) =(-187040)/27
A(1,13) =(-27880)/27
A(1,16) =7750/9
A(1,18) =98371/27
A(1,19) =41027/27
A(1,22) =(-10501)/27
A(1,24) =(-17323)/27
A(1,27) =490/9
A(2,3) =464/3
A(2,4) =86464/27
A(2,5) =15296/3
A(2,6) =55376/27
A(2,8) =6496/27
A(2,11) =(-27880)/27
A(2,12) =(-187040)/27
A(2,13) =(-136424)/27
A(2,15) =(-6896)/9
A(2,18) =41027/27
A(2,19) =98371/27
A(2,21) =7750/9
A(2,24) =(-17323)/27
A(2,26) =(-10501)/27
A(2,30) =490/9
A(3,1) =(-6496)/27
A(3,3) =14176/27
A(3,4) =(-2944)/27
A(3,5) =(-2944)/27
A(3,6) =14176/27
A(3,8) =(-6496)/27
A(3,9) =24784/27
A(3,11) =(-5680)/27
A(3,12) =12736/9
A(3,13) =(-5680)/27
A(3,15) =24784/27
A(3,16) =(-11846)/9
A(3,18) =(-26758)/27
A(3,19) =(-26758)/27
A(3,21) =(-11846)/9
A(3,22) =22789/27
A(3,24) =18262/27
A(3,26) =22789/27
A(3,27) =(-5566)/27
A(3,30) =(-5566)/27
A(3,36) =1
A(4,1) =(-400)/9
A(4,3) =(-3080)/9
A(4,4) =-816
A(4,5) =(-4496)/9
A(4,6) =(-232)/9
A(4,9) =416/3
A(4,11) =7448/9
A(4,12) =9872/9
A(4,13) =1472/9
A(4,16) =(-457)/3
A(4,18) =(-1177)/2
A(4,19) =(-4337)/18
A(4,22) =607/9
A(4,24) =619/6
A(4,27) =(-28)/3
A(5,2) =(-896)/9
A(5,3) =(-392)/9
A(5,4) =2384/9
A(5,5) =2512/9
A(5,6) =232/9
A(5,10) =2336/9
A(5,11) =(-8)/9
A(5,12) =(-3920)/9
A(5,13) =(-976)/9
A(5,17) =(-2104)/9
A(5,18) =123/2
A(5,19) =1981/18
A(5,23) =766/9
A(5,24) =(-307)/18
A(5,28) =(-31)/3
A(6,3) =232/9
A(6,4) =2512/9
A(6,5) =2384/9
A(6,6) =(-392)/9
A(6,7) =(-896)/9
A(6,11) =(-976)/9
A(6,12) =(-3920)/9
A(6,13) =(-8)/9
A(6,14) =2336/9
A(6,18) =1981/18
A(6,19) =123/2
A(6,20) =(-2104)/9
A(6,24) =(-307)/18
A(6,25) =766/9
A(6,29) =(-31)/3
A(7,3) =(-232)/9
A(7,4) =(-4496)/9
A(7,5) =-816
A(7,6) =(-3080)/9
A(7,8) =(-400)/9
A(7,11) =1472/9
A(7,12) =9872/9
A(7,13) =7448/9
A(7,15) =416/3
A(7,18) =(-4337)/18
A(7,19) =(-1177)/2
A(7,21) =(-457)/3
A(7,24) =619/6
A(7,26) =607/9
A(7,30) =(-28)/3
A(8,1) =(-400)/9
A(8,3) =1456/9
A(8,4) =(-992)/9
A(8,5) =(-5248)/9
A(8,6) =(-4096)/9
A(8,7) =(-896)/9
A(8,9) =1552/9
A(8,11) =(-464)/3
A(8,12) =2464/3
A(8,13) =3424/3
A(8,14) =3040/9
A(8,16) =(-761)/3
A(8,18) =(-1831)/9
A(8,19) =-938
A(8,20) =(-1288)/3
A(8,22) =1528/9
A(8,24) =2266/9
A(8,25) =2210/9
A(8,27) =(-406)/9
A(8,29) =(-499)/9
A(8,34) =1
A(9,2) =(-896)/9
A(9,3) =(-4096)/9
A(9,4) =(-5248)/9
A(9,5) =(-992)/9
A(9,6) =1456/9
A(9,8) =(-400)/9
A(9,10) =3040/9
A(9,11) =3424/3
A(9,12) =2464/3
A(9,13) =(-464)/3
A(9,15) =1552/9
A(9,17) =(-1288)/3
A(9,18) =-938
A(9,19) =(-1831)/9
A(9,21) =(-761)/3
A(9,23) =2210/9
A(9,24) =2266/9
A(9,26) =1528/9
A(9,28) =(-499)/9
A(9,30) =(-406)/9
A(9,35) =1
A(10,1) =8/3
A(10,3) =52/3
A(10,4) =40
A(10,5) =24
A(10,6) =4/3
A(10,9) =-8
A(10,11)=(-124)/3
A(10,12)=(-160)/3
A(10,13)=-8
A(10,16)=17/2
A(10,18)=349/12
A(10,19)=47/4
A(10,22)=(-11)/3
A(10,24)=(-61)/12
A(10,27)=1/2
A(11,2) =32/3
A(11,3) =8/3
A(11,4) =(-80)/3
A(11,5) =(-80)/3
A(11,6) =(-8)/3
A(11,10)=(-80)/3
A(11,11)=8/3
A(11,12)=128/3
A(11,13)=32/3
A(11,17)=70/3
A(11,18)=(-43)/6
A(11,19)=(-65)/6
A(11,23)=(-25)/3
A(11,24)=11/6
A(11,28)=1
A(12,3) =4
A(12,4) =40/3
A(12,5) =40/3
A(12,6) =4/3
A(12,11)=(-28)/3
A(12,12)=(-64)/3
A(12,13)=(-16)/3
A(12,18)=29/4
A(12,19)=65/12
A(12,24)=(-17)/12
A(13,3) =4/3
A(13,4) =40/3
A(13,5) =40/3
A(13,6) =4
A(13,11)=(-16)/3
A(13,12)=(-64)/3
A(13,13)=(-28)/3
A(13,18)=65/12
A(13,19)=29/4
A(13,24)=(-17)/12
A(14,3) =(-8)/3
A(14,4) =(-80)/3
A(14,5) =(-80)/3
A(14,6) =8/3
A(14,7) =32/3
A(14,11)=32/3
A(14,12)=128/3
A(14,13)=8/3
A(14,14)=(-80)/3
A(14,18)=(-65)/6
A(14,19)=(-43)/6
A(14,20)=70/3
A(14,24)=11/6
A(14,25)=(-25)/3
A(14,29)=1
A(15,3) =4/3
A(15,4) =24
A(15,5) =40
A(15,6) =52/3
A(15,8) =8/3
A(15,11)=-8
A(15,12)=(-160)/3
A(15,13)=(-124)/3
A(15,15)=-8
A(15,18)=47/4
A(15,19)=349/12
A(15,21)=17/2
A(15,24)=(-61)/12
A(15,26)=(-11)/3
A(15,30)=1/2
A(16,1) =(-8)/3
A(16,3) =56/3
A(16,4) =80/3
A(16,5) =32/3
A(16,9) =32/3
A(16,11)=-40
A(16,12)=-32
A(16,13)=(-8)/3
A(16,16)=(-33)/2
A(16,18)=51/2
A(16,19)=19/3
A(16,22)=73/6
A(16,24)=(-25)/6
A(16,27)=(-25)/6
A(16,31)=1/2
A(17,2) =(-32)/3
A(17,3) =(-160)/3
A(17,4) =(-320)/3
A(17,5) =(-320)/3
A(17,6) =(-160)/3
A(17,7) =(-32)/3
A(17,10)=112/3
A(17,11)=448/3
A(17,12)=224
A(17,13)=448/3
A(17,14)=112/3
A(17,17)=-50
A(17,18)=-150
A(17,19)=-150
A(17,20)=-50
A(17,23)=95/3
A(17,24)=190/3
A(17,25)=95/3
A(17,28)=(-28)/3
A(17,29)=(-28)/3
A(17,32)=1
A(18,4) =32/3
A(18,5) =80/3
A(18,6) =56/3
A(18,8) =(-8)/3
A(18,11)=(-8)/3
A(18,12)=-32
A(18,13)=-40
A(18,15)=32/3
A(18,18)=19/3
A(18,19)=51/2
A(18,21)=(-33)/2
A(18,24)=(-25)/6
A(18,26)=73/6
A(18,30)=(-25)/6
A(18,33)=1/2
A(19,4) =256
A(19,5) =768
A(19,6) =768
A(19,7) =256
A(19,11)=-64
A(19,12)=-896
A(19,13)=-1600
A(19,14)=-768
A(19,18)=176
A(19,19)=992
A(19,20)=816
A(19,24)=-160
A(19,25)=-352
A(19,29)=48
A(20,2) =256
A(20,3) =768
A(20,4) =768
A(20,5) =256
A(20,10)=-768
A(20,11)=-1600
A(20,12)=-896
A(20,13)=-64
A(20,17)=816
A(20,18)=992
A(20,19)=176
A(20,23)=-352
A(20,24)=-160
A(20,28)=48
A(21,4) =128*Sqrt(2)
A(21,5) =128*Sqrt(2)
A(21,11)=-32*Sqrt(2)
A(21,12)=-192*Sqrt(2)
A(21,13)=-32*Sqrt(2)
A(21,18)=40*Sqrt(2)
A(21,19)=40*Sqrt(2)
A(21,24)=-8*Sqrt(2)
A(22,4) =(-65536)/27
A(22,5) =(-114688)/27
A(22,6) =(-40960)/27
A(22,8) =(-8192)/27
A(22,11)=16384/27
A(22,12)=53248/9
A(22,13)=38912/9
A(22,15)=26624/27
A(22,18)=(-32768)/27
A(22,19)=(-10240)/3
A(22,21)=(-10240)/9
A(22,24)=16384/27
A(22,26)=14336/27
A(22,30)=(-2048)/27
A(23,3) =53248/27
A(23,4) =40960/9
A(23,5) =69632/27
A(23,6) =(-8192)/27
A(23,8) =8192/27
A(23,11)=(-48128)/9
A(23,12)=(-22528)/3
A(23,13)=-1024
A(23,15)=(-10240)/9
A(23,18)=14336/3
A(23,19)=8192/3
A(23,21)=14336/9
A(23,24)=(-37888)/27
A(23,26)=(-26624)/27
A(23,30)=2048/9
A(24,1) =8192/27
A(24,3) =(-8192)/27
A(24,4) =69632/27
A(24,5) =40960/9
A(24,6) =53248/27
A(24,9) =(-10240)/9
A(24,11)=-1024
A(24,12)=(-22528)/3
A(24,13)=(-48128)/9
A(24,16)=14336/9
A(24,18)=8192/3
A(24,19)=14336/3
A(24,22)=(-26624)/27
A(24,24)=(-37888)/27
A(24,27)=2048/9
A(25,1) =(-8192)/27
A(25,3) =(-40960)/27
A(25,4) =(-114688)/27
A(25,5) =(-65536)/27
A(25,9) =26624/27
A(25,11)=38912/9
A(25,12)=53248/9
A(25,13)=16384/27
A(25,16)=(-10240)/9
A(25,18)=(-10240)/3
A(25,19)=(-32768)/27
A(25,22)=14336/27
A(25,24)=16384/27
A(25,27)=(-2048)/27
A(26,3) =(-71680)/27
A(26,4) =(-145408)/27
A(26,5) =(-88064)/27
A(26,6) =(-2048)/9
A(26,11)=150016/27
A(26,12)=183296/27
A(26,13)=31232/27
A(26,18)=(-94208)/27
A(26,19)=(-40960)/27
A(26,24)=15872/27
A(27,3) =(-2048)/9
A(27,4) =(-88064)/27
A(27,5) =(-145408)/27
A(27,6) =(-71680)/27
A(27,11)=31232/27
A(27,12)=183296/27
A(27,13)=150016/27
A(27,18)=(-40960)/27
A(27,19)=(-94208)/27
A(27,24)=15872/27
A(28,5) =(-2048)/9
A(28,6) =(-4096)/9
A(28,7) =(-2048)/9
A(28,12)=512/3
A(28,13)=7168/9
A(28,14)=5632/9
A(28,18)=(-256)/9
A(28,19)=(-3584)/9
A(28,20)=(-1792)/3
A(28,24)=512/9
A(28,25)=2048/9
A(28,29)=(-256)/9
A(29,3) =(-2048)/9
A(29,4) =(-8192)/9
A(29,5) =(-4096)/3
A(29,6) =(-8192)/9
A(29,7) =(-2048)/9
A(29,11)=6656/9
A(29,12)=6656/3
A(29,13)=6656/3
A(29,14)=6656/9
A(29,18)=(-7936)/9
A(29,19)=(-15872)/9
A(29,20)=(-7936)/9
A(29,24)=4096/9
A(29,25)=4096/9
A(29,29)=(-256)/3
A(30,2) =(-2048)/9
A(30,3) =(-8192)/9
A(30,4) =(-4096)/3
A(30,5) =(-8192)/9
A(30,6) =(-2048)/9
A(30,10)=6656/9
A(30,11)=6656/3
A(30,12)=6656/3
A(30,13)=6656/9
A(30,17)=(-7936)/9
A(30,18)=(-15872)/9
A(30,19)=(-7936)/9
A(30,23)=4096/9
A(30,24)=4096/9
A(30,28)=(-256)/3
A(31,2) =(-2048)/9
A(31,3) =(-4096)/9
A(31,4) =(-2048)/9
A(31,10)=5632/9
A(31,11)=7168/9
A(31,12)=512/3
A(31,17)=(-1792)/3
A(31,18)=(-3584)/9
A(31,19)=(-256)/9
A(31,23)=2048/9
A(31,24)=512/9
A(31,28)=(-256)/9
A(32,3) =1024*Sqrt(2)/9
A(32,4) =1024*Sqrt(2)/9
A(32,11)=-1792*Sqrt(2)/9
A(32,12)=-256*Sqrt(2)/3
A(32,18)=896*Sqrt(2)/9
A(32,19)=128*Sqrt(2)/9
A(32,24)=-128*Sqrt(2)/9
A(33,5) =1024*Sqrt(2)/9
A(33,6) =1024*Sqrt(2)/9
A(33,12)=-256*Sqrt(2)/3
A(33,13)=-1792*Sqrt(2)/9
A(33,18)=128*Sqrt(2)/9
A(33,19)=896*Sqrt(2)/9
A(33,24)=-128*Sqrt(2)/9
A(34,3) =4096
A(34,4) =8192
A(34,5) =4096
A(34,11)=-9216
A(34,12)=-10240
A(34,13)=-1024
A(34,18)=6144
A(34,19)=2048
A(34,24)=-1024
A(35,4) =4096
A(35,5) =8192
A(35,6) =4096
A(35,11)=-1024
A(35,12)=-10240
A(35,13)=-9216
A(35,18)=2048
A(35,19)=6144
A(35,24)=-1024
A(36,3) =-4096
A(36,4) =-12288
A(36,5) =-12288
A(36,6) =-4096
A(36,11)=11264
A(36,12)=22528
A(36,13)=11264
A(36,18)=-10240
A(36,19)=-10240
A(36,24)=3072
```

Figure A.2 : Matrix $[A]_{36\times 36}$

A(1,1)=1084509/250
A(1,3)=62720001/1000
A(1,4)=148009599/1000
A(1,5)=4098681/50
A(1,6)=(-4220667)/500
A(1,7)=(-3886299)/1000
A(1,8)=1239057/1000
A(1,11)=(-2228553)/125
A(1,13)=(-205859151)/1000
A(1,14)=(-162127899)/500
A(1,15)=(-85727727)/1000
A(1,16)=1337229/125
A(1,17)=(-1224963)/500
A(1,20)=29055969/1000
A(1,22)=994544973/4000
A(1,23)=184869999/800
A(1,24)=92584809/4000
A(1,25)=5508243/4000
A(1,28)=(-23717313)/1000
A(1,30)=(-33656553)/250
A(1,31)=(-61872849)/1000
A(1,32)=(-6293133)/2000
A(1,35)=401433/40
A(1,37)=25352583/800
A(1,38)=22167939/4000
A(1,41)=(-2032683)/1000
A(1,43)=(-256161)/100
A(1,46)=14959/100
A(2,3)=1239057/1000
A(2,4)=(-3886299)/1000
A(2,5)=(-4220667)/500
A(2,6)=4098681/50
A(2,7)=148009599/1000
A(2,8)=62720001/1000
A(2,10)=1084509/250
A(2,13)=(-1224963)/500
A(2,14)=1337229/125
A(2,15)=(-85727727)/1000
A(2,16)=(-162127899)/500
A(2,17)=(-205859151)/1000
A(2,19)=(-2228553)/125
A(2,22)=5508243/4000
A(2,23)=92584809/4000
A(2,24)=184869999/800
A(2,25)=994544973/4000
A(2,27)=29055969/1000
A(2,30)=(-6293133)/2000
A(2,31)=(-61872849)/1000
A(2,32)=(-33656553)/250
A(2,34)=(-23717313)/1000
A(2,37)=22167939/4000
A(2,38)=25352583/800
A(2,40)=401433/40
A(2,43)=(-256161)/100
A(2,45)=(-2032683)/1000
A(2,49)=14959/100
A(3,1)=(-1084509)/250
A(3,3)=7246017/250
A(3,4)=(-9056367)/500
A(3,5)=(-80108109)/500
A(3,6)=(-80108109)/500
A(3,7)=(-9056367)/500
A(3,8)=7246017/250
A(3,10)=(-1084509)/250
A(3,11)=212139/10
A(3,13)=(-61872903)/1000
A(3,14)=45764919/250
A(3,15)=223718679/500
A(3,16)=45764919/250
A(3,17)=(-61872903)/1000
A(3,19)=212139/10
A(3,20)=(-42597873)/1000
A(3,22)=27182547/2000
A(3,23)=(-681986169)/2000
A(3,24)=(-681986169)/2000
A(3,25)=27182547/2000
A(3,27)=(-42597873)/1000
A(3,28)=22436811/500
A(3,30)=51382269/1000
A(3,31)=108928341/500
A(3,32)=51382269/1000
A(3,34)=22436811/500
A(3,35)=(-3263511)/125
A(3,37)=(-81036633)/2000
A(3,38)=(-81036633)/2000
A(3,40)=(-3263511)/125
A(3,41)=3974259/500
A(3,43)=8433609/1000
A(3,45)=3974259/500
A(3,46)=(-993043)/1000
A(3,49)=(-993043)/1000
A(3,55)=1
A(4,1)=(-11907)/25
A(4,3)=(-646623)/100
A(4,4)=(-1491777)/100
A(4,5)=(-40743)/5
A(4,6)=41391/50
A(4,7)=39609/100
A(4,8)=(-12879)/100
A(4,11)=48438/25
A(4,13)=2108673/100
A(4,14)=1629477/50
A(4,15)=853821/100
A(4,16)=(-53703)/50
A(4,17)=6399/25
A(4,20)=(-312687)/100
A(4,22)=(-10136979)/400
A(4,23)=(-1855557)/80
A(4,24)=(-919827)/400
A(4,25)=(-59229)/400
A(4,28)=252999/100
A(4,30)=1367451/100
A(4,31)=1240929/200
A(4,32)=31797/100
A(4,35)=(-4251)/4
A(4,37)=(-1284763)/400
A(4,38)=(-44513)/80
A(4,41)=21409/100
A(4,43)=51869/200
A(4,46)=(-157)/10
A(5,2)=(-25434)/25
A(5,3)=(-56781)/100
A(5,4)=(-50301)/100
A(5,5)=(-107406)/25
A(5,6)=(-108621)/25
A(5,7)=(-39609)/100
A(5,8)=12879/100
A(5,12)=90639/25
A(5,13)=37584/25
A(5,14)=298647/100
A(5,15)=445581/50
A(5,16)=283257/100
A(5,17)=(-6399)/25
A(5,21)=(-10215)/2
A(5,22)=(-687627)/400
A(5,23)=(-1640007)/400
A(5,24)=(-1602207)/400
A(5,25)=(-18927)/400
A(5,29)=72297/20
A(5,30)=8379/8
A(5,31)=63837/40
A(5,32)=12231/40
A(5,36)=(-67117)/50
A(5,37)=(-115031)/400
A(5,38)=(-55323)/400
A(5,42)=24449/100
A(5,43)=593/25
A(5,47)=(-167)/10
A(6,3)=12879/100
A(6,4)=(-39609)/100
A(6,5)=(-108621)/25
A(6,6)=(-107406)/25
A(6,7)=(-50301)/100
A(6,8)=(-56781)/100
A(6,9)=(-25434)/25
A(6,13)=(-6399)/25
A(6,14)=283257/100
A(6,15)=445581/50
A(6,16)=298647/100
A(6,17)=37584/25
A(6,18)=90639/25
A(6,22)=(-18927)/400
A(6,23)=(-1602207)/400
A(6,24)=(-1640007)/400
A(6,25)=(-687627)/400
A(6,26)=(-10215)/2
A(6,30)=12231/40
A(6,31)=63837/40
A(6,32)=8379/8
A(6,33)=72297/20
A(6,37)=(-55323)/400
A(6,38)=(-115031)/400
A(6,39)=(-67117)/50
A(6,43)=593/25
A(6,44)=24449/100
A(6,48)=(-167)/10
A(7,3)=(-12879)/100
A(7,4)=39609/100
A(7,5)=41391/50
A(7,6)=(-40743)/5
A(7,7)=(-1491777)/100
A(7,8)=(-646623)/100
A(7,10)=(-11907)/25
A(7,13)=6399/25
A(7,14)=(-53703)/50
A(7,15)=853821/100
A(7,16)=1629477/50
A(7,17)=2108673/100
A(7,19)=48438/25
A(7,22)=(-59229)/400
A(7,23)=(-919827)/400
A(7,24)=(-1855557)/80
A(7,25)=(-10136979)/400
A(7,27)=(-312687)/100
A(7,30)=31797/100
A(7,31)=1240929/200
A(7,32)=1367451/100
A(7,34)=252999/100
A(7,37)=(-44513)/80
A(7,38)=(-1284763)/400
A(7,40)=(-4251)/4
A(7,43)=51869/200
A(7,45)=21409/100
A(7,49)=(-157)/10
A(8,1)=(-11907)/25
A(8,3)=106191/25
A(8,4)=148959/50
A(8,5)=(-320679)/25
A(8,6)=(-1257849)/50
A(8,7)=(-470934)/25
A(8,8)=(-168318)/25
A(8,9)=(-25434)/25
A(8,11)=2349
A(8,13)=(-1121769)/100
A(8,14)=654399/100
A(8,15)=5075217/100
A(8,16)=5612409/100
A(8,17)=632448/25
A(8,18)=112833/25
A(8,20)=(-477279)/100
A(8,22)=1666503/200
A(8,23)=(-1395351)/50
A(8,24)=(-12094803)/200
A(8,25)=(-3747951)/100
A(8,26)=(-410733)/50
A(8,28)=255753/50
A(8,30)=225423/200
A(8,31)=1351359/50
A(8,32)=5469291/200
A(8,33)=157311/20
A(8,35)=(-76431)/25
A(8,37)=(-744541)/200
A(8,38)=(-490793)/50
A(8,39)=(-420119)/100
A(8,41)=48757/50
A(8,43)=276743/200
A(8,44)=120393/100
A(8,46)=(-13389)/100
A(8,48)=(-15059)/100
A(8,53)=1
A(9,2)=(-25434)/25
A(9,3)=(-168318)/25
A(9,4)=(-470934)/25
A(9,5)=(-1257849)/50
A(9,6)=(-320679)/25
A(9,7)=148959/50
A(9,8)=106191/25
A(9,10)=(-11907)/25
A(9,12)=112833/25
A(9,13)=632448/25
A(9,14)=5612409/100
A(9,15)=5075217/100
A(9,16)=654399/100
A(9,17)=(-1121769)/100
A(9,19)=2349
A(9,21)=(-410733)/50
A(9,22)=(-3747951)/100
A(9,23)=(-12094803)/200
A(9,24)=(-1395351)/50
A(9,25)=1666503/200
A(9,27)=(-477279)/100
A(9,29)=157311/20
A(9,30)=5469291/200
A(9,31)=1351359/50
A(9,32)=225423/200
A(9,34)=255753/50
A(9,36)=(-420119)/100
A(9,37)=(-490793)/50
A(9,38)=(-744541)/200
A(9,40)=(-76431)/25
A(9,42)=120393/100
A(9,43)=276743/200
A(9,45)=48757/50
A(9,47)=(-15059)/100
A(9,49)=(-13389)/100
A(9,54)=1
A(10,1)=81/5
A(10,3)=3969/20
A(10,4)=8991/20
A(10,5)=243
A(10,6)=(-243)/10
A(10,7)=(-243)/20
A(10,8)=81/20
A(10,11)=(-324)/5
A(10,13)=(-12879)/20
A(10,14)=(-9801)/10
A(10,15)=(-5103)/20
A(10,16)=162/5
A(10,17)=(-81)/10
A(10,20)=2061/20
A(10,22)=61677/80
A(10,23)=11151/16
A(10,24)=5481/80
A(10,25)=387/80
A(10,28)=(-1647)/20
A(10,30)=(-4149)/10
A(10,31)=(-1863)/10
A(10,32)=(-387)/40
A(10,35)=137/4
A(10,37)=7783/80
A(10,38)=1339/80
A(10,41)=(-137)/20
A(10,43)=(-157)/20
A(10,46)=1/2
A(11,2)=324/5
A(11,3)=243/10
A(11,4)=243/10
A(11,5)=1296/5
A(11,6)=1296/5
A(11,7)=243/10
A(11,8)=(-81)/10
A(11,12)=(-1134)/5
A(11,13)=(-324)/5
A(11,14)=(-1701)/10
A(11,15)=(-2673)/5
A(11,16)=(-1701)/10
A(11,17)=81/5
A(11,21)=315
A(11,22)=3141/40
A(11,23)=9621/40
A(11,24)=9621/40
A(11,25)=81/40
A(11,29)=(-441)/2
A(11,30)=(-207)/4
A(11,31)=(-189)/2
A(11,32)=-18
A(11,36)=406/5
A(11,37)=599/40
A(11,38)=65/8
A(11,42)=(-147)/10
A(11,43)=(-5)/4
A(11,47)=1
A(12,3)=81/4
A(12,4)=(-243)/20
A(12,5)=(-648)/5
A(12,6)=(-648)/5
A(12,7)=(-243)/20
A(12,8)=81/20
A(12,13)=(-243)/5
A(12,14)=1701/20
A(12,15)=2673/10
A(12,16)=1701/20
A(12,17)=(-81)/10
A(12,22)=2979/80
A(12,23)=(-9621)/80
A(12,24)=(-9621)/80
A(12,25)=(-81)/80
A(12,30)=(-63)/8
A(12,31)=189/4
A(12,32)=9
A(12,37)=(-51)/80
A(12,38)=(-65)/16
A(12,43)=1/8
A(13,3)=81/20
A(13,4)=(-243)/20
A(13,5)=(-648)/5
A(23,6)=(-648)/5
A(13,7)=(-243)/20
A(13,8)=81/4
A(13,13)=(-81)/10
A(13,14)=1701/20
A(13,15)=2673/10
A(13,16)=1701/20
A(13,17)=(-243)/5
A(13,22)=(-81)/80
A(13,23)=(-9621)/80
A(13,24)=(-9621)/80
A(13,25)=2979/80
A(13,30)=9
A(13,31)=189/4
A(13,32)=(-63)/8
A(13,37)=(-65)/16
A(13,38)=(-51)/80
A(13,43)=1/8
A(14,3)=(-81)/10
A(14,4)=243/10
A(14,5)=1296/5
A(14,6)=1296/5
A(14,7)=243/10
A(14,8)=243/10
A(14,9)=324/5
A(14,13)=81/5
A(14,14)=(-1701)/10
A(14,15)=(-2673)/5
A(14,16)=(-1701)/10
A(14,17)=(-324)/5
A(14,18)=(-1134)/5
A(14,22)=81/40
A(14,23)=9621/40
A(14,24)=9621/40
A(14,25)=3141/40
A(14,26)=315
A(14,30)=-18
A(14,31)=(-189)/2
A(14,32)=(-207)/4
A(14,33)=(-441)/2
A(14,37)=65/8
A(14,38)=599/40
A(14,39)=406/5
A(14,43)=(-5)/4
A(14,44)=(-147)/10
A(14,48)=1
A(15,3)=81/20
A(15,4)=(-243)/20
A(15,5)=(-243)/10
A(15,6)=243
A(15,7)=8991/20
A(15,8)=3969/20
A(15,10)=81/5
A(15,13)=(-81)/10
A(15,14)=162/5
A(15,15)=(-5103)/20
A(15,16)=(-9801)/10
A(15,17)=(-12879)/20
A(15,19)=(-324)/5
A(15,22)=387/80
A(15,23)=5481/80
A(15,24)=11151/16
A(15,25)=61677/80
A(15,27)=2061/20
A(15,30)=(-387)/40
A(15,31)=(-1863)/10
A(15,32)=(-4149)/10
A(15,34)=(-1647)/20
A(15,37)=1339/80
A(15,38)=7783/80
A(15,40)=137/4
A(15,43)=(-157)/20
A(15,45)=(-137)/20
A(15,49)=1/2
A(16,1)=(-81)/5
A(16,3)=1053/5
A(16,4)=4617/10
A(16,5)=1863/5
A(16,6)=1053/10
A(16,11)=81
A(16,13)=(-13527)/20
A(16,14)=(-21303)/20
A(16,15)=(-10449)/20
A(16,16)=(-1053)/20
A(16,20)=(-3357)/20
A(16,22)=32409/40
A(16,23)=4086/5
A(16,24)=7551/40
A(16,25)=117/20
A(16,28)=927/5
A(16,30)=(-17631)/40
A(16,31)=(-4671)/20
A(16,32)=(-747)/40
A(16,35)=(-583)/5
A(16,37)=4191/40
A(16,38)=104/5
A(16,41)=411/10
A(16,43)=(-339)/40
A(16,46)=(-147)/20
A(16,50)=1/2
A(17,2)=(-324)/5
A(17,3)=(-2268)/5
A(17,4)=(-6804)/5
A(17,5)=-2268
A(17,6)=-2268
A(17,7)=(-6804)/5
A(17,8)=(-2268)/5
A(17,9)=(-324)/5
A(17,12)=1458/5
A(17,13)=8748/5
A(17,14)=4374
A(17,15)=5832
A(17,16)=4374
A(17,17)=8748/5
A(17,18)=1458/5
A(17,21)=(-2709)/5
A(17,22)=-2709
A(17,23)=-5418
A(17,24)=-5418
A(17,25)=-2709
A(17,26)=(-2709)/5
A(17,29)=1071/2
A(17,30)=2142
A(17,31)=3213
A(17,32)=2142
A(17,33)=1071/2
A(17,36)=(-3017)/10
A(17,37)=(-9051)/10
A(17,38)=(-9051)/10
A(17,39)=(-3017)/10
A(17,42)=959/10
A(17,43)=959/5
A(17,44)=959/10
A(17,47)=(-157)/10
A(17,48)=(-157)/10
A(17,51)=1
A(18,5)=1053/10
A(18,6)=1863/5
A(18,7)=4617/10
A(18,8)=1053/5
A(18,10)=(-81)/5
A(18,14)=(-1053)/20
A(18,15)=(-10449)/20
A(18,16)=(-21303)/20
A(18,17)=(-13527)/20
A(18,19)=81
A(18,22)=117/20
A(18,23)=7551/40
A(18,24)=4086/5
A(18,25)=32409/40
A(18,27)=(-3357)/20
A(18,30)=(-747)/40
A(18,31)=(-4671)/20

Figure A.3$_1$: Matrix $[A^*]_{55\times 55}$

A(18,32)=(-17631)/40
A(18,34)=927/5
A(18,37)=104/5
A(18,38)=4191/40
A(18,40)=(-583)/5
A(18,43)=(-339)/40
A(18,45)=411/10
A(18,49)=(-147)/20
A(18,52)=1/2
A(19,5) =-5184
A(19,6) =-20736
A(19,7) =-31104
A(19,8) =-20736
A(19,9) =-5184
A(19,14)=2592
A(19,15)=28512
A(19,16)=69984
A(19,17)=64800
A(19,18)=20736
A(19,22)=-288
A(19,23)=-10224
A(19,24)=-52560
A(19,25)=-75600
A(19,26)=-32976
A(19,30)=1008
A(19,31)=14832
A(19,32)=40176
A(19,33)=26352
A(19,37)=-1312
A(19,38)=-9392
A(19,39)=-10960
A(19,43)=752
A(19,44)=2192
A(19,48)=-160
A(20,2) =-5184
A(20,3) =-20736
A(20,4) =-31104
A(20,5) =-20736
A(20,6) =-5184
A(20,12)=20736
A(20,13)=64800
A(20,14)=69984
A(20,15)=28512
A(20,16)=2592
A(20,21)=-32976
A(20,22)=-75600
A(20,23)=-52560
A(20,24)=-10224
A(20,25)=-288
A(20,29)=26352
A(20,30)=40176
A(20,31)=14832
A(20,32)=1008
A(20,36)=-10960
A(20,37)=-9392
A(20,38)=-1312
A(20,42)=2192
A(20,43)=752
A(20,47)=-160
A(21,5) =2592*Sqrt(2)
A(21,6) =2592*Sqrt(2)
A(21,14)=-1296*Sqrt(2)
A(21,15)=-5184*Sqrt(2)
A(21,16)=-1296*Sqrt(2)
A(21,22)=144*Sqrt(2)
A(21,23)=2088*Sqrt(2)
A(21,24)=2088*Sqrt(2)
A(21,25)=144*Sqrt(2)
A(21,30)=-216*Sqrt(2)
A(21,31)=-792*Sqrt(2)
A(21,32)=-216*Sqrt(2)
A(21,37)=80*Sqrt(2)
A(21,38)=80*Sqrt(2)
A(21,43)=-8*Sqrt(2)
A(22,5) =(-8188128)/125
A(22,6) =(-28973376)/125
A(22,7) =(-31702752)/125
A(22,8) =(-10077696)/125
A(22,10)=(-839808)/125
A(22,14)=4094064/125
A(22,15)=40625712/125
A(22,16)=75827664/125
A(22,17)=35796816/125
A(22,19)=139968/5
A(22,22)=(-454896)/125
A(22,23)=(-14679144)/125
A(22,24)=(-59463072)/125
A(22,25)=(-46381896)/125
A(22,27)=(-5785344)/125
A(22,30)=1452168/125
A(22,31)=17188848/125
A(22,32)=26693064/125
A(22,34)=4790016/125
A(22,37)=(-1539648)/125
A(22,38)=(-6572664)/125
A(22,40)=(-2052864)/125
A(22,43)=542376/125
A(22,45)=419904/125
A(22,49)=(-31104)/125
A(23,5) =255879/4
A(23,6) =347733/2
A(23,7) =597051/4
A(23,8) =72171/2
A(23,10)=6561/2
A(23,14)=(-255879)/8
A(23,15)=(-2119203)/8
A(23,16)=(-3112101)/8
A(23,17)=(-1135053)/8
A(23,19)=(-28431)/2
A(23,22)=28431/8
A(23,23)=1578285/16
A(23,24)=329508
A(23,25)=3227283/16
A(23,27)=196101/8
A(23,30)=(-158193)/16
A(23,31)=(-795825)/8
A(23,32)=(-2012769)/16
A(23,34)=(-84807)/4
A(23,37)=18225/2
A(23,38)=522693/16
A(23,40)=9477
A(23,43)=(-44469)/16
A(23,45)=(-8019)/4
A(23,49)=1215/8
A(24,4) =-61236
A(24,5) =(-863865)/4
A(24,6) =(-535815)/2
A(24,7) =(-505197)/4
A(24,8) =(-19683)/2
A(24,10)=(-6561)/2
A(24,13)=10206
A(24,14)=1971945/8
A(24,15)=4680909/8
A(24,16)=3483891/8
A(24,17)=570807/8
A(24,19)=15309
A(24,22)=(-280665)/8
A(24,23)=(-5725323)/16
A(24,24)=(-1011609)/2
A(24,25)=(-2422953)/16
A(24,27)=(-231093)/8
A(24,30)=714015/16
A(24,31)=1773981/8
A(24,32)=2139453/16
A(24,34)=223317/8
A(24,37)=(-49653)/2
A(24,38)=(-781893)/16
A(24,40)=(-114453)/8
A(24,43)=81243/16
A(24,45)=28431/8
A(24,49)=(-1215)/4
A(25,3) =5213808/125
A(25,4) =29183328/125
A(25,5) =58471632/125
A(25,6) =49408704/125
A(25,7) =12387168/125
A(25,8) =(-3359232)/125
A(25,10)=839808/125
A(25,13)=(-21893328)/125
A(25,14)=(-96099696)/125
A(25,15)=(-136492128)/125
A(25,16)=(-63090576)/125
A(25,17)=3254256/125
A(25,19)=(-4059072)/125
A(25,22)=36398484/125
A(25,23)=115928496/125
A(25,24)=98365428/125
A(25,25)=10285704/125
A(25,27)=8024832/125
A(25,30)=(-29893212)/125
A(25,31)=(-59819472)/125
A(25,32)=(-18911556)/125
A(25,34)=(-8273664)/125
A(25,37)=12095244/125
A(25,38)=10651824/125
A(25,40)=186624/5
A(25,43)=(-1920996)/125
A(25,45)=(-1353024)/125
A(25,49)=31104/25
A(26,1) =839808/125
A(26,3) =(-3359232)/125
A(26,4) =12387168/125
A(26,5) =49408704/125
A(26,6) =58471632/125
A(26,7) =29183328/125
A(26,8) =5213808/125
A(26,11)=(-4059072)/125
A(26,13)=3254256/125
A(26,14)=(-63090576)/125
A(26,15)=(-136492128)/125
A(26,16)=(-96099696)/125
A(26,17)=(-21893328)/125
A(26,20)=8024832/125
A(26,22)=10285704/125
A(26,23)=98365428/125
A(26,24)=115928496/125
A(26,25)=36398484/125
A(26,28)=(-8273664)/125
A(26,30)=(-18911556)/125
A(26,31)=(-59819472)/125
A(26,32)=(-29893212)/125
A(26,35)=186624/5
A(26,37)=10651824/125
A(26,38)=12095244/125
A(26,41)=(-1353024)/125
A(26,43)=(-1920996)/125
A(26,46)=31104/25
A(27,1) =(-6561)/2
A(27,3) =(-19683)/2
A(27,4) =(-505197)/4
A(27,5) =(-535815)/2
A(27,6) =(-863865)/4
A(27,7) =-61236
A(27,11)=15309
A(27,13)=570807/8
A(27,14)=3483891/8
A(27,15)=4680909/8
A(27,16)=1971945/8
A(27,17)=10206
A(27,20)=(-231093)/8
A(27,22)=(-2422953)/16
A(27,23)=(-1011609)/2
A(27,24)=(-5725323)/16
A(27,25)=(-280665)/8
A(27,28)=223317/8
A(27,30)=2139453/16
A(27,31)=1773981/8
A(27,32)=714015/16
A(27,35)=(-114453)/8
A(27,37)=(-781893)/16
A(27,38)=(-49653)/2
A(27,41)=28431/8
A(27,43)=81243/16
A(27,46)=(-1215)/4
A(28,1) =6561/2
A(28,3) =72171/2
A(28,4) =597051/4
A(28,5) =347733/2
A(28,6) =255879/4
A(28,11)=(-28431)/2
A(28,13)=(-1135053)/8
A(28,14)=(-3112101)/8
A(28,15)=(-2119203)/8
A(28,16)=(-255879)/8
A(28,20)=196101/8
A(28,22)=3227283/16
A(28,23)=329508
A(28,24)=1578285/16
A(28,25)=28431/8
A(28,28)=(-84807)/4
A(28,30)=(-2012769)/16
A(28,31)=(-795825)/8
A(28,32)=(-158193)/16
A(28,35)=9477
A(28,37)=522693/16
A(28,38)=18225/2
A(28,41)=(-8019)/4
A(28,43)=(-44469)/16
A(28,46)=1215/8
A(29,1) =(-839808)/125
A(29,3) =(-10077696)/125
A(29,4) =(-31702752)/125
A(29,5) =(-28973376)/125
A(29,6) =(-8188128)/125
A(29,11)=139968/5
A(29,13)=35796816/125
A(29,14)=75827664/125
A(29,15)=40625712/125
A(29,16)=4094064/125
A(29,20)=(-5785344)/125
A(29,22)=(-46381896)/125
A(29,23)=(-59463072)/125
A(29,24)=(-14679144)/125
A(29,25)=(-454896)/125
A(29,28)=4790016/125
A(29,30)=26693064/125
A(29,31)=17188848/125
A(29,32)=1452168/125
A(29,35)=(-2052864)/125
A(29,37)=(-6572664)/125
A(29,38)=(-1539648)/125
A(29,41)=419904/125
A(29,43)=542376/125
A(29,46)=(-31104)/125
A(30,3) =(-6071112)/125
A(30,4) =(-192456)/5
A(30,5) =9552816/125
A(30,6) =10252656/125
A(30,7) =857304/125
A(30,8) =(-52488)/25
A(30,13)=653184/5
A(30,14)=3242592/125
A(30,15)=(-20435328)/125
A(30,16)=(-6555168)/125
A(30,17)=513216/125
A(30,22)=(-15723558)/125
A(30,23)=3546342/125
A(30,24)=9193662/125
A(30,25)=188082/125
A(30,30)=263736/5
A(30,31)=-18144
A(30,32)=(-147096)/25
A(30,37)=(-1212246)/125
A(30,38)=212706/125
A(30,43)=83916/125
A(31,3) =(-32805)/8
A(31,4) =(-618921)/8
A(31,5) =-148716
A(31,6) =-85293
A(31,7) =(-41553)/8
A(31,8) =10935/8
A(31,13)=92583/4
A(31,14)=1661391/8
A(31,15)=955719/4
A(31,16)=410427/8
A(31,17)=(-5103)/2
A(31,22)=(-1285227)/32
A(31,23)=(-5839047)/32
A(31,24)=(-3059127)/32
A(31,25)=(-94527)/32
A(31,30)=227529/8
A(31,31)=57429
A(31,32)=136971/16
A(31,37)=(-256527)/32
A(31,38)=(-163701)/32
A(31,43)=11097/16
A(32,3) =10935/8
A(32,4) =(-41553)/8
A(32,5) =-85293
A(32,6) =-148716
A(32,7) =(-618921)/8
A(32,8) =(-32805)/8
A(32,13)=(-5103)/2
A(32,14)=410427/8
A(32,15)=955719/4
A(32,16)=1661391/8
A(32,17)=92583/4
A(32,22)=(-94527)/32
A(32,23)=(-3059127)/32
A(32,24)=(-5839047)/32
A(32,25)=(-1285227)/32
A(32,30)=136971/16
A(32,31)=57429
A(32,32)=227529/8
A(32,37)=(-163701)/32
A(32,38)=(-256527)/32
A(32,43)=11097/16
A(33,3) =(-52488)/25
A(33,4) =857304/125
A(33,5) =10252656/125
A(33,6) =9552816/125
A(33,7) =(-192456)/5
A(33,8) =(-6071112)/125
A(33,13)=513216/125
A(33,14)=(-6555168)/125
A(33,15)=(-20435328)/125
A(33,16)=3242592/125
A(33,17)=653184/5
A(33,22)=188082/125
A(33,23)=9193662/125
A(33,24)=3546342/125
A(33,25)=(-15723558)/125
A(33,30)=(-147096)/25
A(33,31)=-18144
A(33,32)=263736/5
A(33,37)=212706/125
A(33,38)=(-1212246)/125
A(33,43)=83916/125
A(34,7) =(-69984)/25
A(34,8) =(-139968)/25
A(34,9) =(-69984)/25
A(34,16)=23328/5
A(34,17)=373248/25
A(34,18)=256608/25
A(34,24)=(-13608)/5
A(34,25)=(-73872)/5
A(34,26)=(-371304)/25
A(34,31)=648
A(34,32)=33696/5
A(34,33)=53784/5
A(34,37)=(-1296)/25
A(34,38)=(-34992)/25
A(34,39)=(-101736)/25
A(34,43)=2592/25
A(34,44)=18792/25
A(34,48)=(-1296)/25
A(35,6) =4374
A(35,7) =13122
A(35,8) =13122
A(35,9) =4374
A(35,15)=-4374
A(35,16)=-25515
A(35,17)=-37908
A(35,18)=-16767
A(35,23)=2673/2
A(35,24)=32805/2
A(35,25)=80919/2
A(35,26)=50787/2
A(35,30)=(-243)/2
A(35,31)=(-16605)/4
A(35,32)=-19602
A(35,33)=(-76869)/4
A(35,37)=1377/4
A(35,38)=8505/2
A(35,39)=30213/4
A(35,43)=-324
A(35,44)=(-5751)/4
A(35,48)=405/4

Figure A.3$_2$: Matrix $\lfloor A^{*} \rfloor_{55\times 55}$

```
A(36,4) =4374
A(36,5) =21870
A(36,6) =43740
A(36,7) =43740
A(36,8) =21870
A(36,9) =4374
A(36,13)=-729
A(36,14)=-21141
A(36,15)=-77274
A(36,16)=-112266
A(36,17)=-73629
A(36,18)=-18225
A(36,22)=2916
A(36,23)=78489/2
A(36,24)=200475/2
A(36,25)=188811/2
A(36,26)=60993/2
A(36,30)=(-18387)/4
A(36,31)=(-140697)/4
A(36,32)=(-226233)/4
A(36,33)=(-103923)/4
A(36,37)=3564
A(36,38)=30537/2
A(36,39)=23409/2
A(36,43)=(-5427)/4
A(36,44)=(-10287)/4
A(36,48)=405/2
A(37,3) =(-69984)/25
A(37,4) =(-419904)/25
A(37,5) =(-209952)/5
A(37,6) =(-279936)/5
A(37,7) =(-209952)/5
A(37,8) =(-419904)/25
A(37,9) =(-69984)/25
A(37,13)=303264/25
A(37,14)=303264/5
A(37,15)=606528/5
A(37,16)=606528/5
A(37,17)=303264/5
A(37,18)=303264/25
A(37,22)=-21384
A(37,23)=-85536
A(37,24)=-128304
A(37,25)=-85536
A(37,26)=-21384
A(37,30)=97848/5
A(37,31)=293544/5
A(37,32)=293544/5
A(37,33)=97848/5
A(37,37)=(-244296)/25
A(37,38)=(-488592)/25
A(37,39)=(-244296)/25
A(37,43)=62856/25
A(37,44)=62856/25
A(37,48)=(-1296)/5
A(38,2) =(-69984)/25
A(38,3) =(-419904)/25
A(38,4) =(-209952)/5
A(38,5) =(-279936)/5
A(38,6) =(-209952)/5
A(38,7) =(-419904)/25
A(38,8) =(-69984)/25
A(38,12)=303264/25
A(38,13)=303264/5
A(38,14)=606528/5
A(38,15)=606528/5
A(38,16)=303264/5
A(38,17)=303264/25
A(38,21)=-21384
A(38,22)=-85536
A(38,23)=-128304
A(38,24)=-85536
A(38,25)=-21384
A(38,29)=97848/5
A(38,30)=293544/5
A(38,31)=293544/5
A(38,32)=97848/5
A(38,36)=(-244296)/25
A(38,37)=(-488592)/25
A(38,38)=(-244296)/25
A(38,42)=62856/25
A(38,43)=62856/25
A(38,47)=(-1296)/5
A(39,2) =4374
A(39,3) =21870
A(39,4) =43740
A(39,5) =43740
A(39,6) =21870
A(39,7) =4374
A(39,12)=-18225
A(39,13)=-73629
A(39,14)=-112266
A(39,15)=-77274
A(39,16)=-21141
A(39,17)=-729
A(39,21)=60993/2
A(39,22)=188811/2
A(39,23)=200475/2
A(39,24)=78489/2
A(39,25)=2916
A(39,29)=(-103923)/4
A(39,30)=(-226233)/4
A(39,31)=(-140697)/4
A(39,32)=(-18387)/4
A(39,36)=23409/2
A(39,37)=30537/2
A(39,38)=3564
A(39,42)=(-10287)/4
A(39,43)=(-5427)/4
A(39,47)=405/2
A(40,2) =4374
A(40,3) =13122
A(40,4) =13122
A(40,5) =4374
A(40,12)=-16767
A(40,13)=-37908
A(40,14)=-25515
A(40,15)=-4374
A(40,21)=50787/2
A(40,22)=80919/2
A(40,23)=32805/2
A(40,24)=2673/2
A(40,29)=(-76869)/4
A(40,30)=-19602
A(40,31)=(-16605)/4
A(40,32)=(-243)/2
A(40,36)=30213/4
A(40,37)=8505/2
A(40,38)=1377/4
A(40,42)=(-5751)/4
A(40,43)=-324
A(40,47)=405/4
A(41,2) =(-69984)/25
A(41,3) =(-139968)/25
A(41,4) =(-69984)/25
A(41,12)=256608/25
A(41,13)=373248/25
A(41,14)=23328/5
A(41,21)=(-371304)/25
A(41,22)=(-73872)/5
A(41,23)=(-13608)/5
A(41,29)=53784/5
A(41,30)=33696/5
A(41,31)=648
A(41,36)=(-101736)/25
A(41,37)=(-34992)/25
A(41,38)=(-1296)/25
A(41,42)=18792/25
A(41,43)=2592/25
A(41,47)=(-1296)/25
A(42,3) =34992*Sqrt(2)/25
A(42,4) =34992*Sqrt(2)/25
A(42,13)=-93312*Sqrt(2)/25
A(42,14)=-11664*Sqrt(2)/5
A(42,22)=18468*Sqrt(2)/5
A(42,23)=6804*Sqrt(2)/5
A(42,30)=-8424*Sqrt(2)/5
A(42,31)=-324*Sqrt(2)
A(42,37)=8748*Sqrt(2)/25
A(42,38)=648*Sqrt(2)/25
A(42,43)=-648*Sqrt(2)/25
A(43,4)=2187*Sqrt(2)
A(43,5) =2187*Sqrt(2)
A(43,13)=-729*Sqrt(2)/2
A(43,14)=-9477*Sqrt(2)/2
A(43,15)=-2187*Sqrt(2)
A(43,22)=729*Sqrt(2)
A(43,23)=12879*Sqrt(2)/4
A(43,24)=2673*Sqrt(2)/4
A(43,30)=-3807*Sqrt(2)/8
A(43,31)=-6723*Sqrt(2)/8
A(43,32)=-243*Sqrt(2)/4
A(43,37)=243*Sqrt(2)/2
A(43,38)=567*Sqrt(2)/8
A(43,43)=-81*Sqrt(2)/8
A(44,6) =2187*Sqrt(2)
A(44,7) =2187*Sqrt(2)
A(44,15)=-2187*Sqrt(2)
A(44,16)=-9477*Sqrt(2)/2
A(44,17)=-729*Sqrt(2)/2
A(44,23)=2673*Sqrt(2)/4
A(44,24)=12879*Sqrt(2)/4
A(44,25)=729*Sqrt(2)
A(44,30)=-243*Sqrt(2)/4
A(44,31)=-6723*Sqrt(2)/8
A(44,32)=-3807*Sqrt(2)/8
A(44,37)=567*Sqrt(2)/8
A(44,38)=243*Sqrt(2)/2
A(44,43)=-81*Sqrt(2)/8
A(45,7) =34992*Sqrt(2)/25
A(45,8) =34992*Sqrt(2)/25
A(45,16)=-11664*Sqrt(2)/5
A(45,17)=-93312*Sqrt(2)/25
A(45,24)=6804*Sqrt(2)/5
A(45,25)=18468*Sqrt(2)/5
A(45,31)=-324*Sqrt(2)
A(45,32)=-8424*Sqrt(2)/5
A(45,37)=648*Sqrt(2)/25
A(45,38)=8748*Sqrt(2)/25
A(45,43)=-648*Sqrt(2)/25
A(46,6) =104976
A(46,7) =209952
A(46,8) =104976
A(46,15)=-104976
A(46,16)=-419904
A(46,17)=-314928
A(46,23)=32076
A(46,24)=274104
A(46,25)=347004
A(46,30)=-2916
A(46,31)=-69984
A(46,32)=-172044
A(46,37)=5832
A(46,38)=37908
A(46,43)=-2916
A(47,5) =-139968
A(47,6) =-419904
A(47,7) =-419904
A(47,8) =-139968
A(47,14)=69984
A(47,15)=606528
A(47,16)=1003104
A(47,17)=466560
A(47,22)=-7776
A(47,23)=-221616
A(47,24)=-793152
A(47,25)=-579312
A(47,30)=22032
A(47,31)=230688
A(47,32)=325296
A(47,37)=-20736
A(47,38)=-79056
A(47,43)=6480
A(48,4) =139968
A(48,5) =559872
A(48,6) =839808
A(48,7) =559872
A(48,8) =139968
A(48,13)=-23328
A(48,14)=-583200
A(48,15)=-1609632
A(48,16)=-1562976
A(48,17)=-513216
A(48,22)=81648
A(48,23)=882576
A(48,24)=1520208
A(48,25)=719280
A(48,30)=-106272
A(48,31)=-578016
A(48,32)=-471744
A(48,37)=60912
A(48,38)=138672
A(48,43)=-12960
A(49,3) =-104976
A(49,4) =-524880
A(49,5) =-1049760
A(49,6) =-1049760
A(49,7) =-524880
A(49,8) =-104976
A(49,13)=419904
A(49,14)=1679616
A(49,15)=2519424
A(49,16)=1679616
A(49,17)=419904
A(49,22)=-661932
A(49,23)=-1985796
A(49,24)=-1985796
A(49,25)=-661932
A(49,30)=513216
A(49,31)=1026432
A(49,32)=513216
A(49,37)=-195372
A(49,38)=-195372
A(49,43)=29160
A(50,3) =139968
A(50,4) =559872
A(50,5) =839808
A(50,6) =559872
A(50,7) =139968
A(50,13)=-513216
A(50,14)=-1562976
A(50,15)=-1609632
A(50,16)=-583200
A(50,17)=-23328
A(50,22)=719280
A(50,23)=1520208
A(50,24)=882576
A(50,25)=81648
A(50,30)=-471744
A(50,31)=-578016
A(50,32)=-106272
A(50,37)=138672
A(50,38)=60912
A(50,43)=-12960
A(51,3) =-139968
A(51,4) =-419904
A(51,5) =-419904
A(51,6) =-139968
A(51,13)=466560
A(51,14)=1003104
A(51,15)=606528
A(51,16)=69984
A(51,22)=-579312
A(51,23)=-793152
A(51,24)=-221616
A(51,25)=-7776
A(51,30)=325296
A(51,31)=230688
A(51,32)=22032
A(51,37)=-79056
A(51,38)=-20736
A(51,43)=6480
A(52,3) =104976
A(52,4) =209952
A(52,5) =104976
A(52,13)=-314928
A(52,14)=-419904
A(52,15)=-104976
A(52,22)=347004
A(52,23)=274104
A(52,24)=32076
A(52,30)=-172044
A(52,31)=-69984
A(52,32)=-2916
A(52,37)=37908
A(52,38)=5832
A(52,43)=-2916
A(53,4) =139968
A(53,5) =279936
A(53,6) =139968
A(53,13)=-23328
A(53,14)=-396576
A(53,15)=-443232
A(53,16)=-69984
A(53,22)=58320
A(53,23)=357696
A(53,24)=167184
A(53,25)=7776
A(53,30)=-47952
A(53,31)=-111456
A(53,32)=-16848
A(53,37)=14256
A(53,38)=10368
A(53,43)=-1296
A(54,5) =139968
A(54,6) =279936
A(54,7) =139968
A(54,14)=-69984
A(54,15)=-443232
A(54,16)=-396576
A(54,17)=-23328
A(54,22)=7776
A(54,23)=167184
A(54,24)=357696
A(54,25)=58320
A(54,30)=-16848
A(54,31)=-111456
A(54,32)=-47952
A(54,37)=10368
A(54,38)=14256
A(54,43)=-1296
A(55,4) =-157464
A(55,5) =-472392
A(55,6) =-472392
A(55,7) =-157464
A(55,13)=26244
A(55,14)=551124
A(55,15)=1049760
A(55,16)=551124
A(55,17)=26244
A(55,22)=-78732
A(55,23)=-656100
A(55,24)=-656100
A(55,25)=-78732
A(55,30)=82377
A(55,31)=295974
A(55,32)=82377
A(55,37)=-33534
A(55,38)=-33534
A(55,43)=3645
```

Figure A.3$_3$: Matrix $[A^*]_{55\times 55}$

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